

On Completions, neat atom structures, and omitting types

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Abstract . This paper has a survey character, but it also contains several new results. The paper tries to give a panoramic picture of the recent developments in algebraic logic. We take a long magical tour in algebraic logic starting from classical notions due to Henkin Monk and Tarski like neat embeddings, culminating in presenting sophisticated model theoretic constructions based on graphs, to solve problems on neat reducts.

We investigate several algebraic notions that apply to varieties of boolean algebras with operators in general *BAOs*, like canonicity, atom-canonicity and completions. We also show that in certain significant special cases, when we have a Stone-like representability notion, like in cylindric, relation and polyadic algebras such abstract notions turn out intimately related to more concrete notions, like complete representability, and the existence of weakly but not strongly representable atom structures.

In connection to the multi-dimensional corresponding modal logic, we prove several omitting types theorem for the finite n variable fragments of first order logic, the multi-dimensional modal logic corresponding to CA_n ; the class of cylindric algebras of dimension n .

A novelty that occurs here is that theories could be uncountable. Our constructions depend on deep model-theoretic results of Shelah.

Several results mentioned in [26] without proofs are proved fully here, one such result is: There exists an uncountable atomic algebra in $\mathfrak{Nr}_n \mathbf{CA}_\omega$ that is not completely representable. Another result: If T is an L_n theory (possibly uncountable), where $|T| = \lambda$, λ is a regular cardinal, and T admits elimination of quantifiers, then $< 2^\lambda$ non principal types *maximal* can be omitted.

A central notion, that connects, representations, completions, complete representations for cylindric algebras is that of neat embedding, which is an algebraic counterpart of Henkin constructions, and is a nut cracker in cylindric-like algebras for proving representation results and related results concerning various forms of the amalgamation property for classes of representable algebras. For example, rep-

representable algebras are those algebras that have the neat embedding property, completely representable countable ones, are the atomic algebras that have the complete neat embedding property. We show that countability cannot be omitted which is sharp in view to our omitting types theorem mentioned above. We show that the class $\mathfrak{Nr}_n \mathbf{CA}_\omega$ is pseudo-elementary, not elementary, and its elementary closure is not finitely axiomatizable for $n \geq 3$. We characterize this class by games.

We give two constructions for weakly representable atom structures that is not strongly representable, that are simple variations on existing themes, and using fairly straightforward modifications of constructions of Hirsch and Hodkinson, we show that the latter class is not elementary for any reduct of polyadic algebras containing all cylindrifiers.

We introduce the new notions of strongly neat, weakly neat and very weakly neat atom structures. An α dimensional atom structure is very weakly neat, α an ordinal, if no algebra based on it is in $\mathfrak{Nr}_\alpha \mathbf{CA}_{\alpha+\omega}$; weakly neat if it has at least one algebra based on it that is in $\mathfrak{Nr}_\alpha \mathbf{CA}_{\alpha+\omega}$, and finally strongly neat if every algebra based on it is in $\mathfrak{Nr}_\alpha \mathbf{CA}_{\alpha+\omega}$. We give examples of the first two, show that they are distinct, and further show that the class of weakly neat atom structures is not elementary. This is done for all dimensions > 2 , infinite included. For the third, we show that finite atom structures are strongly neat (in finite dimensions).

Modifying several constructions in the literature, as well as providing new ones, several results on complete representations and completions are presented, answering several questions posed by Robin Hirsch, and Ian Hodkinson, concerning relation algebras, and complete representability of reducts of polyadic algebras.¹

1 Introduction

Atom canonicity, completions, complete representations and omitting types are four notions that could appear at first glimpse unrelated. The first three, are algebraic notions that apply to varieties of Boolean algebras with operators *BAOs*. Omitting types is a metalogical notion that applies to the corresponding multi-modal logic.

Canonicity is one of the most important concepts of modal completeness theory. From an algebraic perspective, canonical models are not abstract oddities, they are precisely the structure one is led to by underlying the idea in Stone's representability theory for Boolean algebras.

The canonical extension of an algebra has universe the power set algebra of the set of ultrafilters; that is its Stone space, and the extra non-Boolean operations induced naturally from the original ones. A variety is canonical if it is closed under taking canonical extensions.

This is typically a *persistence property*. Persistence properties refer to closure of a variety V under passage from a given member in V , to some

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'larger' algebra .

The other persistence property, namely, atom-canonicity, concerns the atom structure $\text{At}\mathfrak{A}$ of an atomic algebra \mathfrak{A} . As the name suggests, $\text{At}\mathfrak{A}$ is a certain relational structure based on the set of atoms of \mathfrak{A} . A variety V is atom-canonical if it contains the complex algebra $\mathfrak{CmAt}(\mathfrak{A})$ whenever it contains \mathfrak{A} . If $\text{At}V$ is the class of all atom structures of atomic algebras in V , then atom-canonicity amounts to the requirement that $\mathfrak{CmAt}V \subseteq V$.

The (canonical) models of a multi-modal logic \mathfrak{L}_V , corresponding to a canonical variety V , are Kripke frames; these are the ultrafilter frames. The atom structures, are special cases, we call these atomic models.

The canonical extension of an algebra is a complete atomic algebra that the original algebra embeds into, however it only preserves finite meets. Another completion is the Dedekind MacNeille completion, also known as the minimal completion. Every completely additive BAO has such a completion. It is uniquely determined by the fact that it is complete and the original algebra is dense in it; hence it preserves all meets existing in the original algebra. The completion is atomic if and only if the original algebra is. The completion and canonical extension of an algebra only coincide when the algebra is finite.

Complete representations has to do with algebras that have a notion of representations involving - using jargon of modal logic - complex algebras of square frames or, using algebraic logic jargon, full set algebras having square units, and also having the Boolean operations of (concrete) intersections and complementation, like relation algebras and cylindric algebras. Unlike atom-canonicity, this notion is semantical. Such representations is a representation that carries existing (possibly infinite) meets to set theoretic intersections.

Atomic representability is also related to the Orey-Henkin omitting types theorem. Let V be a variety of BAO 's which has a notion of representation, like for example CA_n . The variety CA_n corresponds to the syntactical part of L_n first order logic restricted to the first n variables, while RCA_n corresponds to the semantical models

Indeed given an L_n theory T and a model \mathfrak{M} of T , let ϕ^M denote the set of n -ary assignments satisfying ϕ in \mathfrak{M} , notationally $\phi^{\mathfrak{M}} = \{s \in {}^n M : \mathfrak{M} \models \phi[s]\}$. Then $\{\phi^M : \phi \in L\}$ is the universe of a cylindric set algebra Cs_n with the operations of cylindrifiers corresponding to the semantics of existential quabtifiers and diagonal elements to equality. The class of subdirect products of algebras in Cs_n is the class RCA_n .

A set Γ in the language of T is said to be omitted by the model \mathfrak{M} of T , if $\bigcap \phi^{\mathfrak{M}} = \emptyset$. This can be formulated algebraically as follows: Let $\mathfrak{A} \in \mathbf{CA}_n$ and let there be given a family $(X_i : i \in I)$ of subsets of \mathfrak{A} , then there exists an injective homomorphism $f : \mathfrak{A} \rightarrow \wp(V)$, V a disjoint union of cartesian squares, that omits the X_i 's, that is $\bigcap_{x \in X_i} f(x) = 0$.

The Orey-Henkin omitting types theorem says that this always happens if

\mathfrak{A} is countable and locally finite, and when I is countable, and the X'_i contain only finitely many dimensions (free variables). But it is clear that the above algebraic formulation lends itself to other contexts.

Note that if omitting types theorem holds for arbitrary cardinalities, then the atomic representable algebras are completely representable, by finding a representation that omits the non principal types $\{-x : x \in \text{At}\mathfrak{A}\}$. The converse is false. There are easy examples. Some will be provided below.

Now let us try to find a connection between such notions. Consider a variety V of $BAOs$ that is *not* atom-canonical. This means that there is an $\mathfrak{A} \in V$, such that $\mathfrak{CmAt}\mathfrak{A} \notin V$. If V is completely additive, then $\mathfrak{CmAt}\mathfrak{A}$ is the completion of \mathfrak{A} . So V is not closed under completions.

There are significant varieties that are not atom-canonical, like the variety of representable cylindric algebras \mathbf{RCA}_n for $n \geq 3$ and representable relation algebras \mathbf{RRA} . These have a notion of representability involving square units.

Let \mathfrak{A} be an atomic representable such that $\mathfrak{CmAt}\mathfrak{A} \notin \mathbf{RCA}_n$, for $n \geq 3$. Such algebras exist, the first of its kind was constructed by Hodkinson. The term algebra $\mathfrak{ImAt}\mathfrak{A}$, which is the subalgebra of the complex algebra, is contained in \mathfrak{A} , because \mathbf{RCA}_n is completely additive; furthermore, it is representable but it *cannot* have a complete representation, for such a representation would necessarily induce a representation of $\mathfrak{CmAt}\mathfrak{A}$. That is to say, \mathfrak{A} is an example of an atomic theory in the cylindric modal logic of dimension n , but it has no atomic model. In such a context, $\text{At}\mathfrak{A}$ is also an example of a *weakly representable* atom structure that is not *strongly representable*.

A weakly representable atom structure is an atom structure such that there is at least one algebra based on this atom structure that is representable [25]. It is strongly representable, if every algebra based on it is representable. Hodkinson, the first to construct a weakly representable cylindric atom structure that is not strongly representable, used a somewhat complicated construction depending on so called Rainbow constructions. His proof is model-theoretic. In [25] we use the same method to construct such an atom structure for both relation and cylindric algebras. On the one hand, the graph we used substantially simplifies Hodkinson's construction, and, furthermore, we get our result for relation and cylindric algebras in one go.

Hirsch and Hodkinson show that the class of strongly representable atom structures of relation algebras (and cylindric algebras) is not elementary [13]. The construction makes use of the probabilistic method of Erdős to show that there are finite graphs with arbitrarily large chromatic number and girth. In his pioneering paper of 1959, Erdős took a radically new approach to construct such graphs: for each n he defined a probability space on the set of graphs with n vertices, and showed that, for some carefully chosen probability measures, the probability that an n vertex graph has these properties is positive for all large enough n . This approach, now called the *probabilistic method* has since

unfolded into a sophisticated and versatile proof technique, in graph theory and in other branches of discrete mathematics. This method was used first in algebraic logic by Hirsch and Hodkinson to show that the class of strongly representable atom structures of cylindric and relation algebras is not elementary and that varieties of representable relation algebras are barely canonical [31]. But yet again using these methods of Erdős in [32] it is shown that there exist continuum-many canonical equational classes of Boolean algebras with operators that are not generated by the complex algebras of any first-order definable class of relational structures. Using a variant of this construction the authors resolve the long-standing question of Fine, by exhibiting a bimodal logic that is valid in its canonical frames, but is not sound and complete for any first-order definable class of Kripke frames.

There is an ongoing interplay between algebraic logic on the one hand, and model theory and finite combinatorics particularly graph theory, on the other. Monk was the first to use Ramsey's theorems to construct what is known as Monk's algebras, witnessing non finite axiomatizability for the class of representable cylindric algebras. The key idea of the construction of a Monk's algebra is not so hard. Such algebras are finite, hence atomic, more precisely their Boolean reducts are atomic. The atoms are given colours, and cylindrifications and diagonals are defined by stating that monochromatic triangles are inconsistent. If a Monk's algebra has many more atoms than colours, it follows from Ramsey's Theorem that any representation of the algebra must contain a monochromatic triangle, so the algebra is not representable.

Later Monk-like constructions were substantially generalized by Andr eka N emeti [3], Maddux [18], and finally Hirsch and Hodkinson [14]. Constructing algebras from Erdos graphs have proved extremely rewarding [13], [31], [32]. Another construction invented by Robin Hirsch and Ian Hodkinson is the so-called rainbow construction, which is an ingenious technique that has been used to show that several classes are not elementary [16], [12], [8], and was used together with a lifting argument of Hodkinson of construction polyadic algebras from relation algebras to show that it is undecidable whether a finite relation or cylindric algebra is representable. This shows that certain important products modal logics are undecidable. We will use the rainbow construction below to prove the *CA* analogue of a deep result in [8].

Constructing cylindric algebras based on certain models satisfying certain properties like homogeneity, saturation, elimination of quantifiers, using model theory like in [23], will be generalized below to answer a question of Hirsch [8], on relation algebra reducts of cylindric algebras.

Another model theoretic construction of Hodkinson, based on rainbow graphs, considerably simplified in [25] will be further simplified here to prove that several varieties approximating the class of representable cylindric algebras are not closed under completions.

The main new results

- (1) Answering a question of Robin Hirsch in [11] on complete representations for both relation and cylindric algebras using an example in the same paper. This example shows that in the characterization of countable completely representable algebras, both relation and cylindric algebras, the condition of countability is necessary, it *cannot* be omitted.
- (2) Using an example by Andr  ka et al, to show that unlike cylindric and polyadic equality algebras, atomic polyadic algebras, and Pinters substitution algebras, even without cylindrifiers, of dimension 2 may not be completely representable. However, the class of completely representable algebras is not so hard to characterize; it is finitely axiomatizable in first order logic. This is contrary to a current belief for polyadic algebras, and is an answer to a result of Hodkinson for cylindric free Pinter's algebras.
- (3) Using the construction in Andr  ka et al [3], showing that the omitting types theorem fails for finite first order definable extension of first order logic as defined by Tarski and Givant, and further pursued by others, like Maddux and Biro, a result mentioned in the above cited paper without a proof.
- (4) Characterizing the class $\mathfrak{Nr}_n\mathbf{CA}_\omega$ by games, and showing that the class $\mathfrak{Nr}_n\mathbf{CA}_\omega$ is pseudo elementary, and its elementary closure is not finitely axiomatizable.
- (5) Characterizing the class of countable completely representable algebras of infinitely countable dimensions using weak representations (the question remains whether this class is elementary, the Hirsch Hodkinson example depending on a cardinality argument does not work when our units are weak spaces.)
- (6) Giving full proofs to three results mentioned in [26] without proofs, referring to a pre-print, concerning omitting types in uncountable theories using finitely many variables. This is the pre print, expanded, modified and polished containing proofs of these results and much more. The results concerning omitting types depend on deep model-theoretic constructions of Shelah's.
- (7) We show that the class of weakly neat atom structures, as defined in the abstract, is not elementary for every dimension.
- (8) Unlike the cylindric case, we show that atomic polyadic algebras of infinite dimensions are completely representable.

This paper also simplifies existing proofs in the literature, like the proof in [3], concerning complete representations of relation atom structures, having cylindric basis. Some classical results, like Monk's non-finitizability results for relation and cylindric algebras are also re-proved.

In this preprint we only deal with the notion of weakly neat atom structures. A longer preprint contains all other results.

2 Neat reducts, complete representations and games

Next we characterize the class $\mathfrak{Nr}_n \mathbf{CA}_\omega$ using games. Our treatment in this part follows very closely [11]. The essential difference is that we deal with n dimensional networks and composition moves are replaced by cylindrifier moves in the games.

Definition 2.1. Let n be an ordinal. An s word is a finite string of substitutions (s_i^j) , a c word is a finite string of cylindrifications (c_k) . An sc word is a finite string of substitutions and cylindrifications. Any sc word w induces a partial map $\hat{w} : n \rightarrow n$ by

- $\hat{e} = Id$
- $\widehat{w_j^i} = \hat{w} \circ [i|j]$
- $\widehat{wc_i} = \hat{w} \upharpoonright (n \sim \{i\})$

If $\bar{a} \in {}^{<n-1}n$, we write $\mathbf{s}_{\bar{a}}$, or more frequently $\mathbf{s}_{a_0 \dots a_{k-1}}$, where $k = |\bar{a}|$, for an arbitrary chosen sc word w such that $\hat{w} = \bar{a}$. w exists and does not depend on w by [12, definition 5.23 lemma 13.29]. We can, and will assume [12, Lemma 13.29] that $w = sc_{n-1}c_n$. [In the notation of [12, definition 5.23, lemma 13.29], $\widehat{s_{ijk}}$ for example is the function $n \rightarrow n$ taking 0 to i , 1 to j and 2 to k , and fixing all $l \in n \setminus \{i, j, k\}$.] Let δ be a map. Then $\delta[i \rightarrow d]$ is defined as follows. $\delta[i \rightarrow d](x) = \delta(x)$ if $x \neq i$ and $\delta[i \rightarrow d](i) = d$. We write δ_i^j for $\delta[i \rightarrow \delta_j]$.

Definition 2.2. From now on let $2 \leq n < \omega$. Let \mathfrak{C} be an atomic \mathbf{CA}_n . An *atomic network* over \mathfrak{C} is a map

$$N : {}^n\Delta \rightarrow At\mathfrak{C}$$

such that the following hold for each $i, j < n$, $\delta \in {}^n\Delta$ and $d \in \Delta$:

- $N(\delta_j^i) \leq d_{ij}$
- $N(\delta[i \rightarrow d]) \leq c_i N(\delta)$

Note that N can be viewed as a hypergraph with set of nodes Δ and each hyperedge in ${}^\mu\Delta$ is labelled with an atom from \mathfrak{C} . We call such hyperedges atomic hyperedges. We write $\mathbf{nodes}(N)$ for Δ . But it can happen let N stand for the set of nodes as well as for the function and the network itself. Context will help.

Define $x \sim y$ if there exists \bar{z} such that $N(x, y, \bar{z}) \leq \mathbf{d}_{01}$. Define an equivalence relation \sim over the set of all finite sequences over $\mathbf{nodes}(N)$ by $\bar{x} \sim \bar{y}$ iff $|\bar{x}| = |\bar{y}|$ and $x_i \sim y_i$ for all $i < |\bar{x}|$.

(3) A *hypernetwork* $N = (N^a, N^h)$ over \mathcal{C} consists of a network N^a together with a labelling function for hyperlabels $N^h : {}^{<\omega}\mathbf{nodes}(N) \rightarrow \Lambda$ (some arbitrary set of hyperlabels Λ) such that for $\bar{x}, \bar{y} \in {}^{<\omega}\mathbf{nodes}(N)$

IV. $\bar{x} \sim \bar{y} \Rightarrow N^h(\bar{x}) = N^h(\bar{y})$.

If $|\bar{x}| = k \in \mathbf{nats}$ and $N^h(\bar{x}) = \lambda$ then we say that λ is a k -ary hyperlabel. (\bar{x}) is referred to a k -ary hyperedge, or simply a hyperedge. (Note that we have atomic hyperedges and hyperedges) When there is no risk of ambiguity we may drop the superscripts a, h .

The following notation is defined for hypernetworks, but applies equally to networks.

(4) If N is a hypernetwork and S is any set then $N|_S$ is the n -dimensional hypernetwork defined by restricting N to the set of nodes $S \cap \mathbf{nodes}(N)$. For hypernetworks M, N if there is a set S such that $M = N|_S$ then we write $M \subseteq N$. If $N_0 \subseteq N_1 \subseteq \dots$ is a nested sequence of hypernetworks then we let the *limit* $N = \bigcup_{i < \omega} N_i$ be the hypernetwork defined by $\mathbf{nodes}(N) = \bigcup_{i < \omega} \mathbf{nodes}(N_i)$, $N^a(x_0, \dots, x_{n-1}) = N_i^a(x_0, \dots, x_{n-1})$ if $x_0 \dots x_{\mu-1} \in \mathbf{nodes}(N_i)$, and $N^h(\bar{x}) = N_i^h(\bar{x})$ if $\mathbf{rng}(\bar{x}) \subseteq \mathbf{nodes}(N_i)$. This is well-defined since the hypernetworks are nested and since hyperedges $\bar{x} \in {}^{<\omega}\mathbf{nodes}(N)$ are only finitely long.

For hypernetworks M, N and any set S , we write $M \equiv^S N$ if $N|_S = M|_S$. For hypernetworks M, N , and any set S , we write $M \equiv_S N$ if the symmetric difference $\Delta(\mathbf{nodes}(M), \mathbf{nodes}(N)) \subseteq S$ and $M \equiv^{(\mathbf{nodes}(M) \cup \mathbf{nodes}(N)) \setminus S} N$. We write $M \equiv_k N$ for $M \equiv_{\{k\}} N$.

Let N be a network and let θ be any function. The network $N\theta$ is a complete labelled graph with nodes $\theta^{-1}(\mathbf{nodes}(N)) = \{x \in \mathbf{dom}(\theta) : \theta(x) \in \mathbf{nodes}(N)\}$, and labelling defined by $(N\theta)(i_0, \dots, i_{\mu-1}) = N(\theta(i_0), \theta(i_1), \dots, \theta(i_{\mu-1}))$, for $i_0, \dots, i_{\mu-1} \in \theta^{-1}(\mathbf{nodes}(N))$. Similarly, for a hypernetwork $N = (N^a, N^h)$, we define $N\theta$ to be the hypernetwork $(N^a\theta, N^h\theta)$ with hyperlabelling defined by $N^h\theta(x_0, x_1, \dots) = N^h(\theta(x_0), \theta(x_1), \dots)$ for $(x_0, x_1, \dots) \in {}^{<\omega}\theta^{-1}(\mathbf{nodes}(N))$.

Let M, N be hypernetworks. A *partial isomorphism* $\theta : M \rightarrow N$ is a partial map $\theta : \mathbf{nodes}(M) \rightarrow \mathbf{nodes}(N)$ such that for any $i_1 \dots i_{\mu-1} \in \mathbf{dom}(\theta) \subseteq \mathbf{nodes}(M)$ we have $M^a(i_1, \dots, i_{\mu-1}) = N^a(\theta(i_1), \dots, \theta(i_{\mu-1}))$ and for any finite sequence $\bar{x} \in {}^{<\omega}\mathbf{dom}(\theta)$ we have $M^h(\bar{x}) = N^h\theta(\bar{x})$. If $M = N$ we may call θ a partial isomorphism of N .

Definition 2.3. Let $2 \leq n < \omega$. For any \mathbf{CA}_n atom structure α , and $n \leq m \leq \omega$, we define two-player games $F_n^m(\alpha)$, and $H_n(\alpha)$, each with ω rounds, and for $m < \omega$ we define $H_{m,n}(\alpha)$ with n rounds.

- Let $m \leq \omega$. In a play of $F_n^m(\alpha)$ the two players construct a sequence of networks N_0, N_1, \dots where $\mathbf{nodes}(N_i)$ is a finite subset of $m = \{j : j < m\}$, for each i . In the initial round of this game \forall picks any atom $a \in \alpha$ and \exists must play a finite network N_0 with $\mathbf{nodes}(N_0) \subseteq n$, such that $N_0(\bar{d}) = a$ for some $\bar{d} \in {}^\mu \mathbf{nodes}(N_0)$. In a subsequent round of a play of $F_n^m(\alpha)$ \forall can pick a previously played network N an index $l < n$, a “face” $F = \langle f_0, \dots, f_{n-2} \rangle \in {}^{n-2} \mathbf{nodes}(N)$, $k \in m \setminus \{f_0, \dots, f_{n-2}\}$, and an atom $b \in \alpha$ such that $b \leq c_l N(f_0, \dots, f_i, x, \dots, f_{n-2})$. (the choice of x here is arbitrary, as the second part of the definition of an atomic network together with the fact that $c_i(c_i x) = c_i x$ ensures that the right hand side does not depend on x). This move is called a *cylindrifier move* and is denoted $(N, \langle f_0, \dots, f_{n-2} \rangle, k, b, l)$ or simply (N, F, k, b, l) . In order to make a legal response, \exists must play a network $M \supseteq N$ such that $M(f_0, \dots, f_{i-1}, k, f_i, \dots, f_{n-2}) = b$ and $\mathbf{nodes}(M) = \mathbf{nodes}(N) \cup \{k\}$.

\exists wins $F_n^m(\alpha)$ if she responds with a legal move in each of the ω rounds. If she fails to make a legal response in any round then \forall wins.

- Fix some hyperlabel λ_0 . $H_n(\alpha)$ is a game the play of which consists of a sequence of λ_0 -neat hypernetworks N_0, N_1, \dots where $\mathbf{nodes}(N_i)$ is a finite subset of ω , for each $i < \omega$. In the initial round \forall picks $a \in \alpha$ and \exists must play a λ_0 -neat hypernetwork N_0 with nodes contained in μ and $N_0(\bar{d}) = a$ for some nodes $\bar{d} \in {}^\mu N_0$. At a later stage \forall can make any cylindrifier move (N, F, k, b, l) by picking a previously played hypernetwork N and $F \in {}^{n-2} \mathbf{nodes}(N)$, $l < n, k \in \omega \setminus \mathbf{nodes}(N)$ and $b \leq c_l N(f_0, f_{l-1}, x, f_{n-2})$. [In H_n we require that \forall chooses k as a ‘new node’, i.e. not in $\mathbf{nodes}(N)$, whereas in F_n^m for finite m it was necessary to allow \forall to ‘reuse old nodes’. This makes the game easier as far as \forall is concerned.] For a legal response, \exists must play a λ_0 -neat hypernetwork $M \equiv_k N$ where $\mathbf{nodes}(M) = \mathbf{nodes}(N) \cup \{k\}$ and $M(f_0, f_{i-1}, k, f_{n-2}) = b$. Alternatively, \forall can play a *transformation move* by picking a previously played hypernetwork N and a partial, finite surjection $\theta : \omega \rightarrow \mathbf{nodes}(N)$, this move is denoted (N, θ) . \exists must respond with $N\theta$. Finally, \forall can play an *amalgamation move* by picking previously played hypernetworks M, N such that $M \equiv^{\mathbf{nodes}(M) \cap \mathbf{nodes}(N)} N$ and $\mathbf{nodes}(M) \cap \mathbf{nodes}(N) \neq \emptyset$. This move is denoted (M, N) . To make a legal response, \exists must play a λ_0 -neat hypernetwork L extending M and N , where $\mathbf{nodes}(L) = \mathbf{nodes}(M) \cup \mathbf{nodes}(N)$.

Again, \exists wins $H_n(\alpha)$ if she responds legally in each of the ω rounds, otherwise \forall wins.

- For $m < \omega$ the game $H_{m,n}(\alpha)$ is similar to $H_n(\alpha)$ but play ends after m rounds, so a play of $H_{m,n}(\alpha)$ could be

$$N_0, N_1, \dots, N_m$$

If \exists responds legally in each of these m rounds she wins, otherwise \forall wins.

Definition 2.4. For $m \geq 5$ and $\mathcal{C} \in \mathbf{CA}_m$, if $\mathfrak{A} \subseteq \mathfrak{Nr}_n(\mathcal{C})$ is an atomic cylindric algebra and N is an \mathfrak{A} -network then we define $\hat{N} \in \mathcal{C}$ by

$$\hat{N} = \prod_{i_0, \dots, i_{n-1} \in \text{nodes}(N)} s_{i_0, \dots, i_{n-1}} N(i_0 \dots i_{n-1})$$

$\hat{N} \in \mathcal{C}$ depends implicitly on \mathcal{C} .

We write $\mathfrak{A} \subseteq_c \mathfrak{B}$ if $\mathfrak{A} \in S_c\{\mathfrak{B}\}$.

Lemma 2.5. Let $n < m$ and let \mathfrak{A} be an atomic \mathbf{CA}_n , $\mathfrak{A} \subseteq_c \mathfrak{Nr}_n \mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_m$. For all $x \in \mathcal{C} \setminus \{0\}$ and all $i_0, \dots, i_{n-1} < m$ there is $a \in \text{At}(\mathfrak{A})$ such that $s_{i_0 \dots i_{n-1}} a \cdot x \neq 0$.

Proof. We can assume, see definition 2.1, that $s_{i_0, \dots, i_{n-1}}$ consists only of substitutions, since $c_m \dots c_{m-1} \dots c_n x = x$ for every $x \in \mathfrak{A}$. We have s_j^i is a completely additive operator (any i, j), hence $s_{i_0, \dots, i_{n-1}}$ is too (see definition 2.1). So $\sum \{s_{i_0 \dots i_{n-1}} a : a \in \text{At}(\mathfrak{A})\} = s_{i_0 \dots i_{n-1}} \sum \text{At}(\mathfrak{A}) = s_{i_0 \dots i_{n-1}} 1 = 1$, for any $i_0, \dots, i_{n-1} < n$. Let $x \in \mathcal{C} \setminus \{0\}$. It is impossible that $s_{i_0 \dots i_{n-1}} \cdot x = 0$ for all $a \in \text{At}(\mathfrak{A})$ because this would imply that $1 - x$ was an upper bound for $\{s_{i_0 \dots i_{n-1}} a : a \in \text{At}(\mathfrak{A})\}$, contradicting $\sum \{s_{i_0 \dots i_{n-1}} a : a \in \text{At}(\mathfrak{A})\} = 1$. \square

We now prove two Theorems relating neat embeddings to the games we defined:

Theorem 2.6. Let $n < m$, and let \mathfrak{A} be a \mathbf{CA}_m . If $\mathfrak{A} \in S_c \mathfrak{Nr}_n \mathbf{CA}_m$, then \exists has a winning strategy in $F^m(\text{At}\mathfrak{A})$. In particular if \mathfrak{A} is CR then \exists has a winning strategy in $F^\omega(\text{At}\mathfrak{A})$

Proof. If $\mathfrak{A} \subseteq \mathfrak{Nr}_n \mathcal{C}$ for some $\mathcal{C} \in \mathbf{CA}_m$ then \exists always plays hypernetworks N with $\text{nodes}(N) \subseteq n$ such that $\hat{N} \neq 0$. In more detail, in the initial round, let \forall play $a \in \text{At}\mathfrak{A}$. \exists play a network N with $N(0, \dots, n-1) = a$. Then $\hat{N} = a \neq 0$. At a later stage suppose \forall plays the cylindrifier move $(N, \langle f_0, \dots, f_{\mu-2} \rangle, k, b, l)$ by picking a previously played hypernetwork N and $f_i \in \text{nodes}(N)$, $l < \mu$, $k \notin \{f_i : i < n-2\}$, and $b \leq c_l N(f_0, \dots, f_{i-1}, x, f_{n-2})$. Let $\bar{a} = \langle f_0 \dots f_{l-1}, k \dots f_{n-2} \rangle$. Then $c_k \hat{N} \cdot s_{\bar{a}} b \neq 0$. Then there is a network M such that $\widehat{M \cdot c_k \hat{N}} \cdot s_{\bar{a}} b \neq 0$. Hence $M(f_0, \dots, k, f_{n-2}) = b$. \square

Theorem 2.7. *Let α be a countable \mathbf{CA}_n atom structure. If \exists has a winning strategy in $H_n(\alpha)$, then there is a representable cylindric algebra \mathfrak{C} of dimension ω such that $\mathfrak{Nr}_n \mathfrak{C}$ is atomic and $\text{At} \mathfrak{Nr}_n \mathfrak{C} \cong \alpha$.*

Proof. We shall construct a generalized atomic weak set algebra of dimension ω such that the atom structure of its full neat reduct is isomorphic to the given atom structure. Suppose \exists has a winning strategy in $H_n(\alpha)$. Fix some $a \in \alpha$. We can define a nested sequence $N_0 \subseteq N_1 \dots$ of hypernetworks where N_0 is \exists 's response to the initial \forall -move a , requiring that

1. If N_r is in the sequence and $b \leq c_l N_r(\langle f_0, f_{n-2} \rangle \dots, x, f_{n-2})$. then there is $s \geq r$ and $d \in \text{nodes}(N_s)$ such that $N_s(f_0, f_{i-1}, d, f_{n-2}) = b$.
2. If N_r is in the sequence and θ is any partial isomorphism of N_r then there is $s \geq r$ and a partial isomorphism θ^+ of N_s extending θ such that $\text{rng}(\theta^+) \supseteq \text{nodes}(N_r)$.

We can schedule these requirements to extend so that eventually, every requirement gets dealt with. If we are required to find k and $N_{r+1} \supset N_r$ such that $N_{r+1}(f_0, k, f_{n-2}) = b$ then let $k \in \omega \setminus \text{nodes}(N_r)$ where k is the least possible for definiteness, and let N_{r+1} be \exists 's response using her winning strategy, to the \forall -move $N_r, (f_0, \dots, f_{n-1}), k, b, l$. For an extension of type 2, let τ be a partial isomorphism of N_r and let θ be any finite surjection onto a partial isomorphism of N_r such that $\text{dom}(\theta) \cap \text{nodes}(N_r) = \text{dom} \tau$. \exists 's response to \forall 's move (N_r, θ) is necessarily $N\theta$. Let N_{r+1} be her response, using her winning strategy, to the subsequent \forall -move $(N_r, N_r\theta)$.

Now let N_a be the limit of this sequence. This limit is well-defined since the hypernetworks are nested.

Let θ be any finite partial isomorphism of N_a and let X be any finite subset of $\text{nodes}(N_a)$. Since θ, X are finite, there is $i < \omega$ such that $\text{nodes}(N_i) \supseteq X \cup \text{dom}(\theta)$. There is a bijection $\theta^+ \supseteq \theta$ onto $\text{nodes}(N_i)$ and $j \geq i$ such that $N_j \supseteq N_i, N_i\theta^+$. Then θ^+ is a partial isomorphism of N_j and $\text{rng}(\theta^+) = \text{nodes}(N_i) \supseteq X$. Hence, if θ is any finite partial isomorphism of N_a and X is any finite subset of $\text{nodes}(N_a)$ then

$$\exists \text{ a partial isomorphism } \theta^+ \supseteq \theta \text{ of } N_a \text{ where } \text{rng}(\theta^+) \supseteq X \quad (1)$$

and by considering its inverse we can extend a partial isomorphism so as to include an arbitrary finite subset of $\text{nodes}(N_a)$ within its domain. Let L be the signature with one μ -ary predicate symbol (b) for each $b \in \alpha$, and one k -ary predicate symbol (λ) for each k -ary hyperlabel λ .

For fixed $f_a \in {}^\omega \text{nodes}(N_a)$, let $U_a = \{f \in {}^\omega \text{nodes}(N_a) : \{i < \omega : g(i) \neq f_a(i)\} \text{ is finite}\}$. Notice that U_a is weak unit (a set of sequences agreeing cofinitely with a fixed one)

We can make U_a into the base of an L relativized structure \mathcal{N}_a . Satisfiability for L formulas at assignments $f \in U_a$ is defined the usual Tarskian way.

For $b \in \alpha$, $l_0, \dots, l_{\mu-1}, i_0, \dots, i_{k-1} < \omega$, k -ary hyperlabels λ , and all L -formulas ϕ, ψ , let

$$\begin{aligned} \mathcal{N}_a, f \models b(x_{l_0} \dots x_{l_{\mu-1}}) &\iff N_a(f(l_0), \dots, f(l_{\mu-1})) = b \\ \mathcal{N}_a, f \models \lambda(x_{i_0}, \dots, x_{i_{k-1}}) &\iff N_a(f(i_0), \dots, f(i_{k-1})) = \lambda \\ \mathcal{N}_a, f \models \neg \phi &\iff \mathcal{N}_a, f \not\models \phi \\ \mathcal{N}_a, f \models (\phi \vee \psi) &\iff \mathcal{N}_a, f \models \phi \text{ or } \mathcal{N}_a, f \models \psi \\ \mathcal{N}_a, f \models \exists x_i \phi &\iff \mathcal{N}_a, f[i/m] \models \phi, \text{ some } m \in \text{nodes}(N_a) \end{aligned}$$

For any L -formula ϕ , write $\phi^{\mathcal{N}_a}$ for the set of assignments satisfying it; that is $\{f \in {}^\omega \text{nodes}(N_a) : \mathcal{N}_a, f \models \phi\}$. Let $D_a = \{\phi^{\mathcal{N}_a} : \phi \text{ is an } L\text{-formula}\}$. Then this is the universe of the following weak set algebra

$$\mathcal{D}_a = (D_a, \cup, \sim, D_{ij}, C_i)_{i,j < \omega}$$

then $\mathcal{D}_a \in \mathbf{RCA}_\omega$. (Weak set algebras are representable).

Let $\phi(x_{i_0}, x_{i_1}, \dots, x_{i_k})$ be an arbitrary L -formula using only variables belonging to $\{x_{i_0}, \dots, x_{i_k}\}$. Let $f, g \in U_a$ (some $a \in \alpha$) and suppose π is a partial isomorphism of N_a . We can prove by induction over the quantifier depth of ϕ and using (1), that

$$\mathcal{N}_a, f \models \phi \iff \mathcal{N}_a, g \models \phi \quad (2)$$

Let $\mathcal{C} = \prod_{a \in \alpha} D_a$. Then $\mathcal{C} \in \mathbf{RCA}_\omega$, and \mathcal{C} is the desired generalized weak set algebra. Note that unit of \mathcal{C} is the disjoint union of the weak spaces. We set out to prove our claim. We shall show that $\alpha \cong \mathbf{At} \mathfrak{Nr}_n \mathcal{C}$.

An element x of \mathcal{C} has the form $(x_a : a \in \alpha)$, where $x_a \in D_a$. For $b \in \alpha$ let $\pi_b : \mathcal{C} \rightarrow D_b$ be the projection defined by $\pi_b(x_a : a \in \alpha) = x_b$. Conversely, let $\iota_a : D_a \rightarrow \mathcal{C}$ be the embedding defined by $\iota_a(y) = (x_b : b \in \alpha)$, where $x_a = y$ and $x_b = 0$ for $b \neq a$. Evidently $\pi_b(\iota_a(y)) = y$ for $y \in D_b$ and $\pi_b(\iota_a(y)) = 0$ if $a \neq b$.

Suppose $x \in \mathfrak{Nr}_\mu \mathcal{C} \setminus \{0\}$. Since $x \neq 0$, it must have a non-zero component $\pi_a(x) \in D_a$, for some $a \in \alpha$. Say $\emptyset \neq \phi(x_{i_0}, \dots, x_{i_k})^{\mathcal{D}_a} = \pi_a(x)$ for some L -formula $\phi(x_{i_0}, \dots, x_{i_k})$. We have $\phi(x_{i_0}, \dots, x_{i_k})^{\mathcal{D}_a} \in \mathfrak{Nr}_\mu D_a$. Pick $f \in \phi(x_{i_0}, \dots, x_{i_k})^{\mathcal{D}_a}$ and let $b = N_a(f(0), f(1), \dots, f(n-1)) \in \alpha$. We will show that $b(x_0, x_1, \dots, x_{n-1})^{\mathcal{D}_a} \subseteq \phi(x_{i_0}, \dots, x_{i_k})^{\mathcal{D}_a}$. Take any $g \in b(x_0, x_1, \dots, x_{n-1})^{\mathcal{D}_a}$, so $N_a(g(0), g(1), \dots, g(n-1)) = b$. The map $\{(f(0), g(0)), (f(1), g(1)), \dots, (f(n-1), g(n-1))\}$ is a partial isomorphism of N_a . By (1) this extends to a finite partial isomorphism θ of N_a whose domain includes $f(i_0), \dots, f(i_k)$. Let $g' \in U_a$ be defined by

$$g'(i) = \begin{cases} \theta(i) & \text{if } i \in \text{dom}(\theta) \\ g(i) & \text{otherwise} \end{cases}$$

By (2), $\mathcal{N}_a, g' \models \phi(x_{i_0}, \dots, x_{i_k})$. Observe that $g'(0) = \theta(0) = g(0)$ and similarly $g'(n-1) = g(n-1)$, so g is identical to g' over μ and it differs from g' on only a finite set of coordinates. Since $\phi(x_{i_0}, \dots, x_{i_k})^{D_a} \in \mathfrak{Nr}_\mu(\mathcal{C})$ we deduce $\mathcal{N}_a, g \models \phi(x_{i_0}, \dots, x_{i_k})$, so $g \in \phi(x_{i_0}, \dots, x_{i_k})^{D_a}$. This proves that $b(x_0, x_1 \dots x_{\mu-1})^{D_a} \subseteq \phi(x_{i_0}, \dots, x_{i_k})^{D_a} = \pi_a(x)$, and so

$$\iota_a(b(x_0, x_1, \dots, x_{n-1})^{D_a}) \leq \iota_a(\phi(x_{i_0}, \dots, x_{i_k})^{D_a}) \leq x \in \mathcal{C} \setminus \{0\}.$$

Hence every non-zero element x of $\mathfrak{Nr}_n\mathcal{C}$ is above a an atom $\iota_a(b(x_0, x_1 \dots x_{n-1})^{D_a})$ (some $a, b \in \alpha$) of $\mathfrak{Nr}_n\mathcal{C}$. So $\mathfrak{Nr}_n\mathcal{C}$ is atomic and $\alpha \cong \text{At}\mathfrak{Nr}_n\mathcal{C}$ — the isomorphism is $b \mapsto (b(x_0, x_1, \dots, x_{n-1})^{D_a} : a \in A)$. \square

We can use such games to show that for $n \geq 3$, there is a representable $\mathfrak{A} \in \mathbf{CA}_n$ with atom structure α such that \forall can win the game $F^{n+2}(\alpha)$. However \exists has a winning strategy in $H_n(\alpha)$, for any $n < \omega$. It will follow that there a countable cylindric algebra \mathcal{A}' such that $\mathcal{A}' \equiv \mathcal{A}$ and \exists has a winning strategy in $H(\mathcal{A}')$. So let K be any class such that $\mathfrak{Nr}_n\mathbf{CA}_\omega \subseteq K \subseteq S_c\mathfrak{Nr}_n\mathbf{CA}_{n+2}$. \mathcal{A}' must belong to $\mathfrak{Nr}_n(\mathbf{RCA}_\omega)$, hence $\mathcal{A}' \in K$. But $\mathcal{A} \notin K$ and $\mathcal{A} \preceq \mathcal{A}'$. Thus K is not elementary. From this it easily follows that the class of completely representable cylindric algebras is not elementary, and that the class $\mathfrak{Nr}_n\mathbf{CA}_{n+k}$ for any $k \geq 0$ is not elementary either. Furthermore the constructions works for many variants of cylindric algebras like Halmos' polyadic equality algebras and Pinter's substitution algebras. Formally we shall prove:

Theorem 2.8. *Let $3 \leq n < \omega$. Then the following hold:*

- (i) *Any K such that $\mathfrak{Nr}_n\mathbf{CA}_\omega \subseteq K \subseteq S_c\mathfrak{Nr}_n\mathbf{CA}_{n+2}$ is not elementary.*
- (ii) *The inclusions $\mathfrak{Nr}_n\mathbf{CA}_\omega \subseteq S_c\mathfrak{Nr}_n\mathbf{CA}_\omega \subseteq S\mathfrak{Nr}_n\mathbf{CA}_\omega$ are all proper*

Details of the above idea

Fix finite $n > 2$. We use a rainbow construction for cylindric algebras. We shall construct a cylindric atom structure based on graphs. A coloured graph is an undirected irreflexive graph Γ such that every edge of Γ is coloured by a unique edge colour and some $n-1$ tuples have a unique colour too. \mathcal{Z} denotes the set of integers. Let P be the set of partial order preserving functions $f : \mathcal{Z} \rightarrow \mathcal{N}$ with $|\text{dom}(f)| \leq 2$.

The edge colours (or future atoms) are

- greens: \mathbf{g}_i ($i < n-1$), \mathbf{g}_0^i , $i \in \mathcal{Z}$.
- whites : $\mathbf{w}, \mathbf{w}_f : f \in P$

- yellow : y
- black : b
- reds: r_{ij} ($i, j \in \mathcal{N}$),
- shades of yellow : $y_S : S \subseteq_\omega \mathcal{N}$ or $S = \mathcal{N}$

Definition 2.9. Let $i \in \mathcal{Z}$, and let Γ be a coloured graph consisting of n nodes x_0, \dots, x_{n-2}, z . We call Γ an i -cone if $\Gamma(x_0, z) = \mathbf{g}_i^0$ and for every $1 \leq j \leq n-2$ $\Gamma(x_j, z) = \mathbf{g}_j$, and no other edge of Γ is coloured green. (x_0, \dots, x_{n-2}) is called the centre of the cone, z the apex of the cone and i the tint of the cone. We define a class **J** consisting of coloured graphs with the following properties:

- (1) Γ is a complete graph.
- (2) Γ contains no triangles (called forbidden triples) of the following types:

$$(\mathbf{g}, \mathbf{g}', \mathbf{g}^*), (\mathbf{g}_i, \mathbf{g}_i, \mathbf{w}), \quad \text{any } i \in n-1 \quad (3)$$

$$(\mathbf{g}_0^j, \mathbf{y}, \mathbf{w}_f) \quad \text{unless } f \in P, i \in \text{dom}(f) \quad (4)$$

$$(\mathbf{g}_0^j, \mathbf{g}_0^k, \mathbf{w}_0) \quad \text{any } j, k \in \mathcal{Z} \quad (5)$$

$$(\mathbf{g}_0^i, \mathbf{g}_0^j, \mathbf{r}_{kl}) \quad \text{unless } \{(i, k), (j, l)\} \text{ is an order-} \quad (6)$$

preserving partial function $\mathcal{Z} \rightarrow \mathcal{N}$

$$(\mathbf{y}, \mathbf{y}, \mathbf{y}), (\mathbf{y}, \mathbf{y}, \mathbf{b}) \quad (7)$$

$$(\mathbf{r}_{ij}, \mathbf{r}_{j'k'}, \mathbf{r}_{i^*k^*}) \quad \text{unless } i = i^*, j = j' \text{ and } k' = k^* \quad (8)$$

and no other triple of atoms is forbidden.

- (3) If $a_0, \dots, a_{n-2} \in \Gamma$ are distinct, and no edge (a_i, a_j) $i < j < n$ is coloured green, then the sequence (a_0, \dots, a_{n-2}) is coloured a unique shade of yellow. No other $(n-1)$ tuples are coloured shades of yellow.
- (4) If $D = \{d_0, \dots, d_{n-2}, \delta\} \subseteq \Gamma$ and $\Gamma \upharpoonright D$ is an i -cone with apex δ , inducing the order d_0, \dots, d_{n-2} on its base, and the tuple (d_0, \dots, d_{n-2}) is coloured by a unique shade y_S then $i \in S$.

Proof. The proof is very similar to Hirsch's proof for reation algebras, except that we lift the rainbow construction to cylindric algebras. This is highly non-trivial, it was first done in [?]. The idea is to use labelled graphs as atoms; that is algebras will be based on atom structures consisting of labelled graph. Cylindrifiers are stimulated by shades of yellow, which has to do with the n tuples, and their an additional complexity the presence of cones.

We define a cylindric algebra of dimension n . We first specify its atom structure. Let

$$K = \{a : a \text{ is a surjective map from } n \text{ onto some } \Gamma \in \mathbf{J} \text{ with nodes } \Gamma \subseteq \omega\}.$$

We write Γ_a for the element of K for which $a : \alpha \rightarrow \Gamma$ is a surjection. Let $a, b \in K$ define the following equivalence relation: $a \sim b$ if and only if

- $a(i) = a(j)$ and $b(i) = b(j)$
- $\Gamma_a(a(i), a(j)) = \Gamma_b(b(i), b(j))$ whenever defined
- $\Gamma_a(a(k_0) \dots a(k_{n-2})) = \Gamma_b(b(k_0) \dots b(k_{n-1}))$ whenever defined

Let \mathfrak{C} be the set of equivalence classes. Then define

$$[a] \in E_{ij} \text{ iff } a(i) = a(j)$$

$$[a]T_i[b] \text{ iff } a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}.$$

This, as easily checked, defines a \mathbf{CA}_n atom structure. Let $3 \leq n < \omega$. Let \mathcal{C}_n be the complex algebra over \mathfrak{C} . We will show that \mathcal{C}_n is not in $S_c \mathfrak{Nr}_n \mathbf{CA}_{n+2}$ but an elementary extension of \mathcal{A} belongs to $\mathfrak{Nr}_n \mathbf{CA}_\omega$. But first we translate our games to games on coloured graphs: Let N be an atomic \mathcal{C}_n network. Let x, y be two distinct nodes occuring in the n tuple \bar{z} . $N(\bar{z})$ is an atom of \mathcal{C}_n which defines an edge colour of x, y . Using the fact that the dimension is at least 3, the edge colour depends only on x and y not on the other elements of \bar{z} or the positions of x and y in \bar{z} . Similarly N defines shades of white for certain $(n-1)$ tuples. In this way N translates into a coloured graph. This translation has an inverse. More precisely we have: Let $\Gamma \in \mathbf{J}$ be arbitrary. Define N_Γ whose nodes are those of Γ as follows. For each $a_0, \dots, a_{n-1} \in \Gamma$, define $N_\Gamma(a_0, \dots, a_{n-1}) = [\alpha]$ where $\alpha : n \rightarrow \Gamma \upharpoonright \{a_0, \dots, a_{n-1}\}$ is given by $\alpha(i) = a_i$ for all $i < n$. Then, as easily checked, N_Γ is an atomic \mathcal{C}_n network. Conversely, let N be any non empty atomic \mathcal{C}_n network. Define a complete coloured graph Γ_N whose nodes are the nodes of N as follows:

- For all distinct $x, y \in \Gamma_N$ and edge colours η , $\Gamma_N(x, y) = \eta$ if and only if for some $\bar{z} \in {}^n N$, $i, j < n$, and atom $[\alpha]$, we have $N(\bar{z}) = [\alpha]$, $z_i = x$, $z_j = y$ and the edge $(\alpha(i), \alpha(j))$ is coloured η in the graph α .
- For all $x_0, \dots, x_{n-2} \in {}^{n-1} \Gamma_N$ and all yellows y_S , $\Gamma_N(x_0, \dots, x_{n-2}) = y_S$ if and only if for some \bar{z} in ${}^n N$, $i_0, i_{n-2} < n$ and some atom $[\alpha]$, we have $N(\bar{z}) = [\alpha]$, $z_{i_j} = x_j$ for each $j < n-1$ and the $n-1$ tuple $\langle \alpha(i_0), \dots, \alpha(i_{n-2}) \rangle$ is coloured y_S . Then Γ_N is well defined and is in \mathbf{J} .

The following is then, though tedious and long, easy to check: For any $\Gamma \in \mathbf{J}$, we have $\Gamma_{N_\Gamma} = \Gamma$, and for any \mathcal{C}_n network N $N_{\Gamma_N} = N$. This translation makes the following equivalent formulation of the games $F^m(\text{At}\mathcal{C}_n)$, originally defined on networks. The new game builds a nested sequence $\Gamma_0 \subseteq \Gamma_1 \subseteq \dots$ of coloured graphs. Let us start with the game $F^m(\alpha)$. \forall picks a graph $\Gamma_0 \in \mathbf{J}$ with $\Gamma_0 \subseteq m$ and $|\Gamma_0| = m$. \exists makes no response to this move. In a subsequent round, let the last graph built be Γ_i . \forall picks

- a graph $\Phi \in \mathbf{J}$ with $|\Phi| = m$
- a single node $k \in \Phi$
- a coloured graph embedding $\theta : \Phi \sim \{k\} \rightarrow \Gamma_i$. Let $F = \phi \setminus \{k\}$. Then F is called a face. \exists must respond by amalgamating Γ_i and Φ with the embedding θ . In other words she has to define a graph $\Gamma_{i+1} \in C$ and embeddings $\lambda : \Gamma_i \rightarrow \Gamma_{i+1}$ $\mu : \phi \rightarrow \Gamma_{i+1}$, such that $\lambda \circ \theta = \mu \upharpoonright F$.

Let us consider the possibilities. There may be already a point $z \in \Gamma_i$ such that the map $(k \rightarrow z)$ is an isomorphism over F . In this case \exists does not need to extend the graph Γ_i , she can simply let $\Gamma_{i+1} = \Gamma_i$ $\lambda = Id_{\Gamma_i}$, and $\mu \upharpoonright F = Id_F$, $\mu(\alpha) = z$. Otherwise, without loss of generality, let $F \subseteq \Gamma_i$, $k \notin \Gamma_i$. Let Γ_i^* be the colored graph with nodes $\text{nodes}(\Gamma_i) \cup \{k\}$, whose edges are the combined edges of Γ_i and Φ , such that for any $n-1$ tuple \bar{x} of nodes of Γ_i^* , the color $\Gamma_i^*(\bar{x})$ is

- $\Gamma_i(\bar{x})$ if the nodes of \bar{x} all lie in Γ and $\Gamma_i(\bar{x})$ is defined
- $\phi(\bar{x})$ if the nodes of \bar{x} all lie in ϕ and $\phi(\bar{x})$ is defined
- undefined, otherwise.

\exists has to complete the labeling of Γ_i^* by adding all missing edges, colouring each edge (β, k) for $\beta \in \Gamma_i \sim \Phi$ and then choosing a shade of white for every $n-1$ tuple \bar{a} of distinct elements of Γ_i^* not wholly contained in Γ_i nor Φ , if none of the edges in \bar{a} is coloured green. She must do this on such a way that the resulting graph belongs to \mathbf{J} . If she survives each round, \exists has won the play. Notice that \exists has a winning strategy in the in $F^m(\text{At}(\mathfrak{C}_n))$ if and only if and only if she has a winning strategy in the graph games defined above. This is tedious and rather long to verify but basically routine. We now show that the rainbow algebra \mathcal{A} (definition above) is not in $\mathbf{S_c}\mathfrak{Nr}_\mu\mathbf{CA}_{n+2}$.

For that we show \forall can win the game $F^{n+2}(\text{At}(\mathfrak{C}_n))$. In his zeroth move, \forall plays a graph $\Gamma \in \mathbf{J}$ with nodes $0, 1, \dots, n-1$ and such that $\Gamma(i, j) = \mathbf{w}(i < j < n-1)$, $\Gamma(i, n-1) = \mathbf{g}_i(i = 1, \dots, n)$, $\Gamma(0, n-1) = \mathbf{g}_0^0$, and $\Gamma(0, 1, \dots, n-2) = \mathbf{y}_\omega$. This is a 0-cone with base $\{0, \dots, n-2\}$. In the following moves, \forall repeatedly chooses the face $(0, 1, \dots, n-2)$ and demands a node (possibly used before)

α with $\Phi(i, \alpha) = \mathbf{g}_i (i = 1, \dots, n-2)$ and $\Phi(0, \alpha) = \mathbf{g}_0^\alpha$, in the graph notation – i.e., an α -cone on the same base. \exists , among other things, has to colour all the edges connecting nodes. The idea is that by the rules of the game only permissible colours would be red. Using this, \forall can force a win eventually for else we are led to a decreasing sequence in \mathcal{N} . In more detail, In the initial round \forall plays a graph Γ with nodes $0, 1, \dots, n-1$ such that $\Gamma(i, j) = \mathbf{w}$ for $i < j < n-1$ and $\Gamma(i, n-1) = \mathbf{g}_i$ ($i = 1, \dots, n-2$), $\Gamma(0, n-1) = \mathbf{g}_0^0$ and $\Gamma(0, 1 \dots n-2) = \mathbf{y}_N$. \exists must play a graph with $\Gamma_1(0, \dots, n-1) = \mathbf{g}_0$. In the following move \forall chooses the face $(0, \dots, n-2)$ and demands a node n with $\Gamma_2(i, n) = \mathbf{g}_i$ and $\Gamma_2(0, n) = \mathbf{g}_0^{-1}$. \exists must choose a label for the edge $(n, n-1)$ of Γ_2 . It must be a red atom r_{mn} . Since $-1 < 0$ we have $m < n$. In the next move \forall plays the face $(0, \dots, n-2)$ and demands a node $n+1$ such that $\Gamma_3(i, n+1) = \mathbf{g}_i^{-2}$. Then $\Gamma_3(n+1, n)$ and $\Gamma_3(n+1, n-1)$ both being red, the indices must match. $\Gamma_3(n+1, n) = r_{ln}$ and $\Gamma_3(n+1, n-1) = r_{lm}$ with $l < m$. In the next round \forall plays $(0, 1 \dots n-2)$ and reuses the node $n-2$ such that $\Gamma_4(0, n-2) = \mathbf{g}_0^{-3}$. This time we have $\Gamma_4(n, n-1) = r_{jl}$ for some $j < l \in N$. Continuing in this manner leads to a decreasing sequence in \mathcal{N} .

Recall from definition 2.3 that $H_k(\alpha)$ is the hypernetwork game with k rounds. The translation of the games H and H_k to graphs is as follows.

- Fix some hyperlabel λ_0 . $H_\mu(\alpha)$ is a game the play of which consists of a sequence of λ_0 -neat hypernetworks N_0, N_1, \dots where $\mathbf{nodes}(N_i)$ is a finite subset of ω , for each $i < \omega$. A neat hypernetwork, now, is a pair (Γ, N^h) with Γ a coloured graph. N^h are the hyperlabels, these we forget for a while and identify the pair (Γ, N^h) with Γ . Hyperedges will be dealt with later, and we shall see that they are easier to deal with. \forall picks a graph $\Gamma_0 \in \mathbf{J}$ with $\Gamma_0 \subseteq_\omega \omega$ and here we do not require that $|\Gamma_0| = n$. \exists make no response to this move. In a subsequent round, let the last graph built be Γ_i . \forall picks

- a graph $\Phi \in \mathbf{J}$ with $|\Phi| = |\Gamma|$
- a single node $k \in \Phi$
- a colored graph embedding $\theta : \Phi \sim \{k\} \rightarrow \Gamma_i$. Let $F = \Phi \sim \{k\}$. Then F is called a face. \exists must respond by amalgamating Γ_i and Φ with the embedding θ as before. In other words she has to define a graph $\Gamma_{i+1} \in \mathbf{C}$ and embeddings $\lambda : \Gamma_i \rightarrow \Gamma_{i+1}$ $\mu : \Phi \rightarrow \Gamma_{i+1}$, such that $\lambda \circ \theta = \mu \upharpoonright F$.

Now we may write N_Γ or simply N instead of Γ , but in all cases we are dealing with *coloured graphs* that is *the translation* of networks. That is when we write N then, N will be viewed as a coloured graph. Alternatively, \forall can play a *transformation move* by picking a previously played graph N and a partial, finite surjection $\theta : \omega \rightarrow \mathbf{nodes}(N)$, this

move is denoted (N, θ) . \exists must respond with $N\theta$. Finally, \forall can play an *amalgamation move* by picking previously played graphs M, N such that $M \equiv_{\text{nodes}(M) \cap \text{nodes}(N)} N$ and $\text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset$. This move is denoted (M, N) . To make a legal response, \exists must play a λ_0 -neat hypernetwork L extending M and N , where $\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)$. Again, \exists wins $H(\alpha)$ if she responds legally in each of the ω rounds, otherwise \forall wins.

It will simplify things a bit if we alter the rules of the game $H(\alpha)$ slightly. We impose certain restrictions on \forall . First a piece of notation. A triangle move Γ in the graph game will be denoted by (Γ, F, k, b, l) , $\Phi = F \cup \{k\}$ and $b \leq c_l N_\Gamma(f_0, \dots, f_{n-1})$.

- \forall is only allowed to play a triangle move (Γ, F, k, b, l) if there does not exist $l \in \text{nodes}(\Gamma)$ such that $N_\Gamma(f_0, \dots, f_i, l \dots f_{n-2}) = b$.
- \forall is only allowed to play transformation moves (N, θ) if θ is injective.
- \forall is only allowed to play an amalgamation move (M, N) if for all $m \in \text{nodes}(M) \setminus \text{nodes}(N)$ and all $n \in \text{nodes}(N) \setminus \text{nodes}(M)$ the map $\{(m, n)\} \cup \{(x, x) : x \in \text{nodes}(M) \cap \text{nodes}(N)\}$ is not a partial isomorphism. I.e. he can only play (M, N) if the amalgamated part is ‘as large as possible’.

If, as a result of these restrictions, \forall cannot move at some stage then he loses and the game halts.

It is easy to check that \forall has a winning strategy in $H(\alpha)$ iff he has a winning strategy with these restrictions to his moves. Also, if \forall plays with these restrictions to his moves, if \exists has a winning strategy then she has a winning strategy which only directs her to play strict hypernetworks. The same holds when we consider $H_n(\alpha)$. We will assume that \forall plays according to these restrictions.

Now we deal with hypernetworks. Notice that in this game \forall is not allowed to reuse nodes and so his winning strategy above does not work. In a play of $N(\alpha)$ \exists is required to play λ_0 neat hypernetworks, so she has no choice about the hyperedges for short edges, these are labelled by λ_0 . In response to a cylindrifier move (N, F, k, l) all long hyperedges not incident with k necessarily keep the hyperlabel they had in N . All long hyperedges incident with k in N are given unique hyperlabels not occurring as the hyperlabel of any other hyperedge in M . We assume, without loss of generality, that we have infinite supply of hyperlabels of all finite arities so this is possible. In response to an amalgamation move (M, N) all long hyperedges whose range is contained in $\text{nodes}(M)$ have hyperlabel determined by M , and those whose range is contained in nodes N have hyperlabel determined by N . If \bar{x} is a

long hyperedge of \exists 's response L where $\text{rng}(\bar{x}) \not\subseteq \text{nodes}(M)$, $\text{nodes}(N)$ then \bar{x} is given a new hyperlabel, not used in any previously played hypernetwork and not used within L as the label of any hyperedge other than \bar{x} . This completes her strategy for labelling hyperedges.

Now we give \exists 's strategy for edge labelling. We need some notation and terminology. Every irreflexive edge of any hypernetwork has an owner \forall or \exists namely the one who played this edge. We call such edges \forall edges or \exists edges. Each long hyperedge \bar{x} in a hypernetwork N occurring in the play has an envelope $v_N(\bar{x})$ to be defined shortly. In the initial round of \forall plays $a \in \alpha$ and \exists plays N_0 then all irreflexive edges of N_0 belongs to \forall . There are no long hyperedges in N_0 . If in a later move, \forall plays the transformation move (N, θ) and \exists responds with $N\theta$ then owners and envelopes are inherited in the obvious way. If \forall plays a cylindrifier move (N, F, k, b, l) and \exists responds with M then the owner in M of an edge not incident with k is the same as it was in N and the envelope in M of a long hyperedge not incident with k is the same as that it was in N . The edges $(f, k), (k, f)$ belong to \forall in M all edges $(l, k)(k, l)$ for $l \in \text{nodes}(N) \sim F$ belong to \exists in M . if \bar{x} is any long hyperedge of M with $k \in \text{rng}(\bar{x})$, then $v_M(\bar{x}) = \text{nodes}(M)$. If \forall plays the amalgamation move (M, N) and \exists responds with L then for $m \neq n \in \text{nodes}(L)$ the owner in L of a edge (m, n) is \forall if it belongs to \forall in either M or N , in all other cases it belongs to \exists in L . If \bar{x} is a long hyperedge of L then $v_L(\bar{x}) = v_M(x)$ if $\text{range}(x) \subseteq \text{nodes}(M)$ $v_L(x) = v_N(x)$ and $v_L(x) = \text{nodes}(M)$ otherwise. This completes the definition of owners and envelopes. By induction on the number of rounds one can show

Claim . *Let M, N occur in a play of $H_n(\alpha)$ in which \exists uses default labelling for hyperedges. Let \bar{x} be a long hyperedge of M and let \bar{y} be a long hyperedge of N .*

- (1) *For any hyperedge \bar{x}' with $\text{rng}(\bar{x}') \subseteq_M (\bar{x})$, if $M(\bar{x}') = M(\bar{x})$ then $\bar{x}' = \bar{x}$.*
- (2) *if \bar{x} is a long hyperedge of M and \bar{y} is a long hyperedge of N , and $M(\bar{x}) = N(\bar{y})$ then there is a local isomorphism $\theta : v_M(\bar{x}) \rightarrow v_N(\bar{y})$ such that $\theta(x_i) = y_i$ for all $i < |x|$.*
- (3) *For any $x \in \text{nodes}(M) \sim v_M(\bar{x})$ and $S \subseteq v_M(\bar{x})$, if (x, s) belong to \forall in M for all $s \in S$, then $|S| \leq 2$.*

Now we define \exists 's strategy for choosing the labels for edges and yellow colours for $n - 1$ hyperedges. Let N_0, N_1, \dots, N_r be the start of a play of $H_k(\alpha)$ just before round $r + 1$. \exists computes partial functions $\rho_s : Z \rightarrow N$, for $s \leq r$. Inductively for $s \leq r$ suppose

- I. If $N_s(x, y)$ is green or yellow then (x, y) belongs to \forall in N_s .

- II. $\rho_0 \subseteq \dots \rho_r$
- III. $\text{dom}(\rho_s) = \{i \in Z : \exists t \leq s, x, y \in \text{nodes}(N_t), N_t(x, y) = \mathbf{g}_0^i\}$
- IV. ρ_s is order preserving: if $i < j$ then $\rho_s(i) < \rho_s(j)$. The range of ρ_s is widely spaced: if $i < j \in \text{dom}\rho_s$ then $\rho_s(i) - \rho_s(j) \geq 3^{n-r}$, where $n - r$ is the number of rounds remaining in the game.
- V. For $u, v, x, y \in \text{nodes}(N_s)$, if $N_s(u, v) = \mathbf{r}_{\mu, \delta}$, $N_s(x, u) = \mathbf{g}_0^i$, $N_s(x, v) = \mathbf{g}_0^j$, $N_s(y, u) = N_s(y, v) = \mathbf{y}$ then
 - (a) if $N_s(x, y) \neq \mathbf{w}_f$ then $\rho_s(i) = \mu$ and $\rho_s(j) = \delta$
 - (b) If $N_s(x, y) = \mathbf{w}_f$ for some $f \in P$, the $\mu = f(i)$, $\delta = f(j)$.
- VI. N_s is a strict λ_0 neat hypernetwork.

To start with if \forall plays a in the initial round then $\text{nodes}(N_0) = \{0, 1, \dots, n-1\}$, the hyperedge labelling is defined by $N_0(0, 1, \dots, n) = a$.

In response to a triangle move $(N_s, F, k, \mathbf{g}_0^p, l)$ by \forall , for some $s \leq r$ and some $p \in Z$, \exists must extend ρ_r to ρ_{r+1} so that $p \in \text{dom}(\rho_{r+1})$ and the gap between elements of its range is at least 3^{n-r-1} . Inductively ρ_r is order preserving and the gap between its elements is at least 3^{n-r} , so this can be maintained in a further round. If \forall chooses non green atoms, green atoms with the same suffix, or green atom whose suffices already belong to ρ_r , there would be fewer elements to add to the domain of ρ_{r+1} , which makes it easy for \exists to define ρ_{r+1} . This establishes properties II – IV for round $r + 1$.

Let us assume that \forall play the triangle move (N_s, F, k, a, l) in round $r + 1$. \exists has to choose labels for $\{(x, k), (k, x)\}$ $x \in \text{nodes}(N_s) \sim F$. She chooses labels for the edges (x, k) one at a time and then determines the reverse edges (k, x) uniquely. Property I is clear since in all cases the only atoms \exists chooses are white, black or red.

Now we distinguish between two cases.

If x and k are both apexes of cones on F , then \exists has no choice but to pick a red atom. The colour she chooses is uniquely defined. Other wise, this is not the case, so for some $i < n - 1$ there is no $f \in F$ such that $N_s(\beta, f), N_s(f, x)$ are both coloured \mathbf{g}_i or if $i = 0$, they are coloured \mathbf{g}_0^l and $\mathbf{g}_0^{l'}$ for some l and l' .

Now we distinguish between several subcases:

- (1) Suppose that it is not the case that there exists distinct $i, j \in F$ $N_s(x, i)$ and a are both green and $N_s(x, j)$ and a are both green. Let $S = \{p \in Z : (N_s(x, i) = \mathbf{g}_0^p \wedge a = \mathbf{y}) \vee N_s(x, i) = \mathbf{y} \wedge a = \mathbf{g}_0^p)\}$ Then $|S| \leq 2$. \exists lets $N_{s+1}(x, k) = \mathbf{w}_f$ for some f with $\text{dom}(f) = S$.

Suppose that there exists i, j distinct in F such that $N_s(i, j) = \mathbf{r}_{\beta, \mu}$, $N_s(x, i) = \mathbf{g}_p$, $N_s(x, j) = \mathbf{g}_q$ for some $p, q \in Z$. By property (IV) $f = \{(p, \beta), (q, \mu)\}$ is order preserving. \exists lets $N_{s+1}(x, k) = \mathbf{w}_f$ in this case.

In all other cases: either there are $i, j \in F$ such that $N_s(i, j)$ is not red, or if it is then it is not the case that $N_s(x, i)$ $N_s(x, j)$ are both green, and it is not the case that $N_s(x, i) = N_s(x, j) = \mathbf{y}$, \exists lets $f : S \rightarrow N$ an arbitrary

order preserving function. The only forbidden triangles involving \mathbf{w}_f are avoided. Since \exists does not change green or yellow atoms to label new edges and $N_{r+1}(x, k) = \mathbf{w}_f$, all triangles involving the new edge (x, k) are consistent in N_{r+1} . Clearly property VI holds after round $r + 1$.

- (2) Else if it is not the case that $N_s(x, i) = a$ and not the case that $N_s(x, j) = y$, \exists lets $N_r(x, k) = \mathbf{b}$. Property V is not applicable in this case. The only forbidden triple involving the atom \mathbf{b} is avoided, so all triangles (x, y, k) are consistent in N_{r+1} and property VI holds after round $r + 1$ lets $N_{r+1}(x, k) = a$.
- (3) If neither case above applies, then for all distinct $i, j \in F$ either $N_s(x, i) = \mathbf{g}_p$, $a = \mathbf{g}_q$ (for some p and q) and $N_s(x, j) = N_s(x, i) = a = \mathbf{y}$ and $N_s(x, j) = \mathbf{g}_p$, $b = \mathbf{g}_q$. Assume the first alternative. There are two subcases.
 - (i) $N_s(i, j) \neq w_f$ for all $f \in P$. \exists lets $\mu = \rho_{r+1}(p)$, $\delta = \rho_{r+1}(q)$, maintaining property Va . The only forbidden triples of atoms involving $r_{\mu, \delta}$ are avoided. The triple of atoms form a triangle (x, y, k) will not be forbidden since the only green edge incident with k is (i, k) and since ρ_{r+1} is order preserving. To check forbidden triple (16) suppose $N_s(x, y), N_{r+1}(k)$ are both red for some $y \in \text{nodes}(N_r)$. We have $y \notin \{i, j\}$ so \exists chose the red label $N_{r+1}(y, k)$. By her strategy we have $N_s(i, y) = \mathbf{g}_t$ and $N_s(j, y) = \mathbf{y}$. By property Va for N_{r+1} we have $N_{r+1}(x, y) = r_{\rho_{r+1}(p)\rho_{r+1}(t)}$ and $N_{r+1}(y, k) = r$. The property VI holds for N_{r+1} .
 - (ii) $N_s(i, j) = \mathbf{w}_f$ for some $f \in F$. By consistency of N_s and forbidden triple (14), we have $p \in \text{dom}(f)$ and since \forall 's move we have $q \in \text{dom}(f)$. \exists lets $\mu = f(p)$, $\delta = f(q)$ maintaining property V for round $r + 1$. As above, the only forbidden triples of atoms involving $r_{\mu, \delta}$ are (15) and (16). Since f is order preserving and since the only green edge incident with k is (i, k) in N_{r+1} triangles involving the new edge (x, k) cannot give a forbidden triple. For forbidden triple (16) let $y \in \text{nodes}(N_s)$ and suppose $N_{r+1}(x, y)N_{r+1}(y, k)$ are both red. As above, by her strategy we must have $N_s(y, i) = \mathbf{g}_t$ for some t and $N_s(y, j) = \mathbf{y}$. By consistency of N_s we have $t \in \text{dom}(f)$ and the current part of her strategy she lets $N_{r+1}(y, k) = r_{f(t), f(q)}$. By property Vb for N_s we have $N_{r+1}(x, y) = r_{f(p), f(t)}$. So the triple of atoms from the triangle (x, y, k) is not forbidden by (16). This establishes property (VI) for N_{r+1} .

We have finished with triangle moves. Now we move to amalgamation moves. Although our hypernetworks are all strict, it is not necessarily the case that

hyperlabels label unique hyperedges - amalgamation moves can force that the same hyperlabel can label more than one hyperedge. However, within the envelope of a hyperedge \bar{x} , the hyperlabel $N(\bar{x})$ is unique.

We consider an amalgamation move (N_s, N_t) chosen by \forall in round $r + 1$. \exists has to choose a label for each edge (i, j) where $i \in \text{nodes}(N_s) \sim \text{nodes}(N_t)$ and $j \in \text{nodes}(N_t) \sim (N_s)$. This determines the label for the reverse edge. Let \bar{x} enumerate $\text{nodes}(N_s) \cap \text{nodes}(N_t)$. If \bar{x} is short, then there are at most two nodes in the intersection and this case is similar to the triangle move. if not, that is if \bar{x} is long in N_s , then by the claim there is a partial isomorphism $\theta : v_{N_s}(\bar{x}) \rightarrow v_{N_t}(\bar{x})$ fixing \bar{x} . We can assume that $v_{N_s}(\bar{x}) = \text{nodes}(N_s) \cap \text{nodes}(N_t) = \text{rng}(\bar{x}) = v_{N_t}(\bar{x})$. It remains to label the edges $(i, j) \in N_{r+1}$ where $i \in \text{nodes}(N_s) \sim \text{nodes}(N_t)$ and $j \in \text{nodes}(N_t) \sim \text{nodes}(N_s)$. Her strategy is similar to the triangle move. if i and j are tints of the same cone she choose a red. If not she chooses white atom if possible, else the black atom if possible, otherwise a red atom. She never chooses a green atom, she lets $\rho_{r+1} = \rho_r$ and properties *II*, *III*, *IV* remain true in round $r + 1$.

- (1) There is no $x \in \text{nodes}(N_s) \cap \text{nodes}(N_t)$ such that $N_s(i, x)$ and $N_t(x, j)$ are both green. If there are nodes $u, v \in \text{nodes}(N_s) \cap \text{nodes}(N_t)$ such that $N_s(u, v) = r_{\beta, \mu}$, $N_s(i, u) = g_p$, $N_s(i, v) = g_q$, $N_t(u, j) = N_t(v, j) = y$ for some $\beta, \mu \in N$, $p, q \in Z$ or the roles of i, j are swapped, she lets $f = \{(p, \beta), (q, \mu)\}$ and sets $N_{r+1}(i, j) = w_f$. Since all edges labelled by green or yellow atoms belong to \forall , we can apply the above claim to show that the points u, v are unique so f is well defined. This is also true if \bar{x} is short, since in this case there are only two nodes in $\text{nodes}(N_s) \cap \text{nodes}(N_t)$.

If there are no such u, v as described then let $S = \{p \in Z : \exists y \in \text{nodes}(N_s) \cap \text{nodes}(N_t), (N_s(i, y) = g_p \wedge N_t(y, j) = y) \vee (N_s(i, y) = y \wedge N_t(y, j) = g_p)\}$. Then $|S| \leq 2$. Let f be any order preserving function and \exists let $N_{r+1} = w_f$. Property (VI) holds for N_{r+1} as for triangle moves.

- (2) Otherwise if there is no such x , then she lets $N_r(i, j) = b$. As with triangle moves all properties are maintained.
- (3) Otherwise, there are $x, y \in \text{nodes}(N_s) \cap \text{nodes}(N_t)$ such that $N_s(i, x) = g_k$, $N_s(x, j) = g_l$ for some $k, l \in N$ and $N_s(i, y) = N_t(y, j) = y$. By claim 3 x, y are unique. She labels (i, j) in N_r with a red atom $r_{\beta, \mu}$ where
 - (i) If $N_s(x, y) \neq w_f$ for all $f \in P$, then $\beta = \rho_{r+1}(k)$, $\mu = \rho_{r+1}(l)$. This maintains property *Va*.
 - (ii) Otherwise $N_s(x, y) = w_f$ for some $f \in F$ and $\beta = f(k)$ $\mu = f(l)$.

Now we turn to coloring of n -tuples. For each tuple $\bar{a} = a_0, \dots, a_{n-2} \in (\Gamma)^{n-1}$ with no edge (a_i, a_j) coloured green, \exists colours \bar{a} by y_S , where $S = \{i \in N : \text{there is an } i \text{ cone in } \Gamma \text{ with base } a_0, \dots, a_{n-2}\}$. We need to check that such labeling works.

Let us check that $(n-1)$ tuples are labeled correctly, by yellow colours. Let D be set of n nodes, and suppose that $\Gamma \upharpoonright D$ is an i cone with apex δ and base $\{d_0, \dots, d_{n-2}\}$, and that the tuple (d_0, \dots, d_{n-2}) is labelled y_S in Γ . We need to show that $i \in S$. If $D \subseteq \Gamma$, then inductively the graph Γ constructed so far is in **J**, and therefore $i \in S$. If $D \subseteq \Phi$ then as \forall chose Φ in **J** we get also $i \in S$. If neither holds, then D contains α and also some $\beta \in \Gamma \sim \Phi$. \exists chose the colour $\Gamma^+(\alpha, \beta)$ and her strategy ensures her that it is green. Hence neither α or β can be the apex of the cone $\Gamma^+ \upharpoonright D$, so they must both lie in the base \bar{d} . This implies that \bar{d} is not yet labelled in Γ^* , so \exists has applied her strategy to choose the colour y_S to label \bar{d} in Γ^+ . But this strategy will have chosen S containing i since $\Gamma^* \upharpoonright D$ is already a cone in Γ^* . Also \exists never chooses a green edge, so all green edges of Γ^+ lie in Γ^* .

That leaves one (hard) case, where there are two nodes $\beta, \beta' \in \Gamma$, \exists colours both (β, α) and (β', α) red, and the old edge (β, β') has already been coloured red (earlier in the game). If (β, β') was coloured by \exists , then there is no problem. So suppose, for a contradiction, that (β, β') was coloured by \forall . Since \exists chose red colours for (α, β) and (α, β') , it must be the case that there are cones in Γ^* with apexes α, β, β' and the same base, F , each inducing the same linear ordering $\bar{f} = (f_0, \dots, f_{n-2})$, say, on F . Of course, the tints of these cones may all be different. Clearly, no edge in F is labeled green, as no cone base can contain green edges. It follows that \bar{f} must be labeled by some yellow colour, y_S , say. Since $\Phi \in \mathbf{J}$, it obeys its definition, so the tint i (say) of the cone from α to \bar{f} lies in S . Suppose that λ was the last node of $F \cup \{\beta, \beta'\}$ to be created, as the game proceeded. As $|F \cup \{\beta, \beta'\}| = n+1$, we see that \exists must have chosen the colour of at least one edge in this : say, (λ, μ) . Now all edges from β into F are green, and so chosen by \forall , and the edge (β, β') was also chosen by him. The same holds for edges from β' to F . Hence $\lambda, \mu \in F$. We can now see that it was \exists who chose the colour y_S of \bar{f} . For y_S was chosen in the round when F 's last node, i.e., λ was created. It could only have been chosen by \forall if he also picked the colour of every edge in F involving λ . This is not so, as the edge (λ, μ) was coloured by \exists , and lies in F . As $i \in S$, it follows from the definition of \exists 's strategy that at the time when λ was added, there was already an i -cone with base \bar{f} , and apex γ say. We claim that $F \cup \{\alpha\}$ and $F \cup \{\gamma\}$ are isomorphic over F . For this, note that the only $(n-1)$ -tuples of either $F \cup \{\alpha\}$ or $F \cup \{\gamma\}$ with a yellow colour are in F (since all others involve a green edge). But this means that \exists could have taken $\alpha = \gamma$ in the current round, and not extended the graph. This is contrary to our original assumption, and completes the proof.

Let \mathcal{A} be the rainbow algebra defined above. There is a countable cylindric algebra \mathcal{A}' such that $\mathcal{A}' \equiv \mathcal{A}$ and \exists has a winning strategy in $H(\mathcal{A}')$. We have seen that for $n < \omega$ \exists has a winning strategy σ_n in $H_n(\mathcal{A})$. We can assume that σ_n is deterministic. Let \mathcal{B} be a non-principal ultrapower of \mathcal{A} . We can show that \exists has a winning strategy σ in $H(\mathcal{B})$ — essentially she uses σ_n in the n 'th component of the ultraproduct so that at each round of $H(\mathcal{B})$ \exists is still winning in co-finitely many components, this suffices to show she has still not lost. Now use an elementary chain argument to construct countable elementary subalgebras $\mathcal{A} = \mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \dots \preceq \mathcal{B}$. For this, let \mathcal{A}_{i+1} be a countable elementary subalgebra of \mathcal{B} containing \mathcal{A}_i and all elements of \mathcal{B} that σ selects in a play of $H_\omega(\mathcal{B})$ in which \forall only chooses elements from \mathcal{A}_i . Now let $\mathcal{A}' = \bigcup_{i < \omega} \mathcal{A}_i$. This is a countable elementary subalgebra of \mathcal{B} and \exists has a winning strategy in $H(\mathcal{A}')$. ■

3 The class of neat reducts

Here we study the class of neat reducts.

Theorem 3.1. (1) *Let $n > 1$ and $\alpha \geq \omega$. Then the class $\mathfrak{Nr}_n \mathbf{CA}_\alpha = \mathfrak{Nr}_n \mathbf{RCA}_\alpha$ is pseudo-elementary, but is not elementary. Furthermore, $El\mathfrak{Nr}_n \mathbf{CA}_\omega \subset \mathbf{RCA}_n$, $El\mathfrak{Nr}_n \mathbf{CA}_\omega$ is recursively enumerable, and for $n > 2$ is not finitely axiomatizable.*

(2) *The class \mathbf{CCA}_n of dimension $n \geq 3$, is also psuedo elementary and furthermore, its elementary closure is not finitely axiomatizable*

Proof. We first show that for any infinite α , $\mathfrak{Nr}_n \mathbf{CA}_\omega = \mathfrak{Nr}_n \mathbf{CA}_\alpha$. Let $\mathfrak{A} \in \mathfrak{Nr}_n \mathbf{CA}_\omega$, so that $\mathfrak{A} = \mathfrak{Nr}_n \mathfrak{B}'$, $\mathfrak{B}' \in \mathbf{CA}_\omega$. Let $\mathfrak{B} = \mathfrak{S}g^{\mathfrak{B}'} \mathfrak{A}$. Then $\mathfrak{B} \in \mathbf{Lf}_\omega$, and $\mathfrak{A} = \mathfrak{Nr}_n \mathfrak{B}$. But $\mathbf{Lf}_\omega = \mathfrak{Nr}_\omega \mathbf{Lf}_\alpha$ and we are done. To show that $\mathfrak{Nr}_n \mathbf{CA}_\omega \subseteq \mathfrak{Nr}_n \mathbf{RCA}_\omega$, let $\mathfrak{A} \in \mathfrak{Nr}_n \mathbf{CA}_\omega$, then by the above argument there exists then $\mathfrak{B} \in \mathbf{Lf}_\omega$ such that $\mathfrak{A} = \mathfrak{Nr}_n \mathfrak{B}$. by $\mathbf{Lf}_\omega \subseteq \mathbf{RCA}_\omega$, we are done.

Hence It is known that class $\mathfrak{Nr}_n \mathbf{CA}_\omega$ is not elementary. In fact, there is an algebra $\mathfrak{A} \in \mathfrak{Nr}_n \mathbf{CA}_\omega$ having a complete subalgebra \mathfrak{B} , and $\mathfrak{B} \notin \mathfrak{Nr}_n \mathbf{CA}_{n+1}$

To show that it is pseudo-elementary, we use a three sorted defining theory, with one sort for a cylindric algebra of dimension n (c), the second sort for the Boolean reduct of a cylindric algebra (b) and the third sort for a set of dimensions (δ). We use superscripts n, b, δ for variables and functions to indicate that the variable, or the returned value of the function, is of the sort of the cylindric algebra of dimension n , the Boolean part of the cylindric algebra or the dimension set, respectively. The signature includes dimension sort constants i^δ for each $i < \omega$ to represent the dimensions. The defining theory for $\mathfrak{Nr}_n \mathbf{CA}_\omega$ incudes sentences demanding that the consatnts i^δ for $i < \omega$ are

distinct and that the last two sorts define a cylindric algebra of dimension ω . For example the sentence

$$\forall x^\delta, y^\delta, z^\delta (d^b(x^\delta, y^\delta) = c^b(z^\delta, d^b(x^\delta, z^\delta).d^b(z^\delta, y^\delta)))$$

represents the cylindric algebra axiom $\mathbf{d}_{ij} = \mathbf{c}_k(\mathbf{d}_{ik}.\mathbf{d}_{kj})$ for all $i, j, k < \omega$. We have a function I^b from sort c to sort b and sentences requiring that I^b be injective and to respect the n dimensional cylindric operations as follows: for all x^r

$$\begin{aligned} I^b(\mathbf{d}_{ij}) &= d^b(i^\delta, j^\delta) \\ I^b(\mathbf{c}_i x^r) &= \mathbf{c}_i^b(I^b(x)). \end{aligned}$$

Finally we require that I^b maps onto the set of n dimensional elements

$$\forall y^b ((\forall z^\delta (z^\delta \neq 0^\delta, \dots (n-1)^\delta \rightarrow c^b(z^\delta, y^b) = y^b)) \leftrightarrow \exists x^r (y^b = I^b(x^r))).$$

For $\mathfrak{A} \in \mathbf{CA}_n$, $\mathfrak{Ad}_3\mathfrak{A}$ denotes the \mathbf{CA}_3 obtained from \mathfrak{A} by discarding all operations indexed by indices in $n \sim 3$. \mathbf{Df}_n denotes the class of diagonal free cylindric algebras. $\mathfrak{Ad}_{df}\mathfrak{A}$ denotes the \mathbf{Df}_n obtained from \mathfrak{A} by deleting all diagonal elements. To prove the non-finite axiomatizability result we use Monk's algebras. For $3 \leq n, i < \omega$, with $n-1 \leq i$, $\mathfrak{C}_{n,i}$ denotes the \mathbf{CA}_n associated with the cylindric atom structure as defined on p. 95 of [6]. Then by [6, 3.2.79] for $3 \leq n$, and $j < \omega$, $\mathfrak{Ad}_3\mathfrak{C}_{n,n+j}$ can be neatly embedded in a \mathbf{CA}_{3+j+1} . (1) By [6, 3.2.84]) we have for every $j \in \omega$, there is an $3 \leq n$ such that $\mathfrak{Ad}_{df}\mathfrak{Ad}_3\mathfrak{C}_{n,n+j}$ is a non-representable \mathbf{Df}_3 . (2) Now suppose $m \in \omega$. By (2), choose $j \in \omega \sim 3$ so that $\mathfrak{Ad}_{df}\mathfrak{Ad}_3\mathfrak{C}_{j,j+m+n-4}$ is a non-representable \mathbf{Df}_3 . By (1) we have $\mathfrak{Ad}_{df}\mathfrak{Ad}_3\mathfrak{C}_{j,j+m+n-4} \subseteq \mathfrak{Nr}_3\mathfrak{B}_m$, for some $\mathfrak{B} \in \mathbf{CA}_{n+m}$. Put $\mathfrak{A}_m = \mathfrak{Nr}_n\mathfrak{B}_m$. $\mathfrak{Ad}_{df}\mathfrak{A}_m$ is not representable, a friotri, $\mathfrak{A}_m \notin \mathbf{RCA}_n$, for else its \mathbf{Df} reduct would be representable. Therefore $\mathfrak{A}_m \notin \mathbf{ELNr}_n\mathbf{CA}_\omega$. Now let \mathfrak{C}_m be an algebra similar to \mathbf{CA}_ω 's such that $\mathfrak{B}_m = \mathfrak{Ad}_{n+m}\mathfrak{C}_m$. Then $\mathfrak{A}_m = \mathfrak{Nr}_n\mathfrak{C}_m$. Let F be a non-principal ultrafilter on ω . Then

$$\prod_{m \in \omega} \mathfrak{A}_m / F = \prod_{m \in \omega} (\mathfrak{Nr}_n\mathfrak{C}_m) / F = \mathfrak{Nr}_n(\prod_{m \in \omega} \mathfrak{C}_m / F)$$

But $\prod_{m \in \omega} \mathfrak{C}_m / F \in \mathbf{CA}_\omega$. Hence $\mathbf{CA}_n \sim \mathbf{ELNr}_n\mathbf{CA}_\omega$ is not closed under ultraproducts. It follows that the latter class is not finitely axiomatizable. In [20] it is proved that for $1 < \alpha < \beta$, $\mathbf{ELNr}_\alpha\mathbf{CA}_\beta \subset \mathbf{SNr}_\alpha\mathbf{CA}_\beta$. ■

From the above proof it follows that

Corollary 3.2. *Let K be any class such that $\mathfrak{Nr}_n\mathbf{CA}_\omega \subseteq K \subseteq \mathbf{RCA}_n$. Then \mathbf{ELK} is not finitely axiomatizable*

3.1 Weakly but not strongly neat atom structures

Definition 3.3. An atom structure of dimension α is strongly neat, if any algebra having this atom structure is in $\mathfrak{Nr}_\alpha \mathbf{CA}_{\alpha+\omega}$. It is weakly neat if one algebra based on it is neat reduct, very weakly neat if no algebra based on it is a neat reduct.

Since there are representable algebras that are not neat reducts all algebras we are about to construct will be variation on themes of representable algebras constructed in previous publications. For completely representable algebras an atom structure is completely representable if it has at least one algebra having this atom structure that is completely representable. This implies that all algebras based on its atom structure are completely representable.

Here we show that this not the case with neat atom structures. For this pupose, we construct a weakly neat atom structure, that is not strongly neat. Similar constructions were used to show that the class of neat reduct is not closed under forming subalgebras, and that there are isomorphic algebras that generate non -isomorphic ones in extra dimensiona, and that neat subreduct may not be full neat reducts, Such construction, and several related ones, were also used to confirm unresolved a conjecture of Tarski. The construction is joint with Istvan Németi.

We put this construction to another new use, and therefore we find it appropriate to give the details with several variants that appeared in three publications of the author. The latter also shows in addition that there are representable cylindric algebras, satisfying the merry go round identities but cannot be a reduct of a $QEAs$.

Theorem 3.4. *For every ordinal $\alpha > 1$, there exists a weakly neat atom structure of dimension α , that is not strongly neat.*

Proof. Let $\alpha > 1$ and \mathfrak{F} is field of charactteristic 0. Let

$$V = \{s \in {}^\alpha \mathfrak{F} : |\{i \in \alpha : s_i \neq 0\}| < \omega\},$$

Note that V is a vector space over the field \mathfrak{F} . We will show that V is a weakly neat atom structure that is not strongly neat. Indeed V is a concrete atom structure $\{s\} \equiv_i \{t\}$ if $s(j) = t(j)$ for all $j \neq i$, and $\{s\} \equiv_{ij} \{t\}$ if $s \circ [i, j] = t$. Let y denote the following α -ary relation:

$$y = \{s \in V : s_0 + 1 = \sum_{i>0} s_i\}.$$

Note that the sum on the right hand side is a finite one, since only finitely many of the s_i 's involved are non-zero. For each $s \in y$, we let y_s be the singleton containing s , i.e. $y_s = \{s\}$. Define $\mathfrak{A} \in WQEAs_\alpha$ as follows:

$$\mathfrak{A} = \mathfrak{Sg}^c\{y, y_s : s \in y\}.$$

We shall prove that

$$\mathfrak{Ad}_{SC}\mathfrak{A} \notin \mathfrak{Nr}_\alpha SC_{\alpha+1}.$$

That is for no $\mathfrak{P} \in SC_{\alpha+1}$, it is the case that $\mathfrak{S}g^e X$ exhausts the set of all α dimensional elements of \mathfrak{P} . So assume, seeking a contradiction, that $\mathfrak{Ad}_{SC}\mathfrak{A} \in \mathfrak{Nr}_\alpha SC_{\alpha+1}$. Let $X = \{y_s : s \in y\}$. Of course every element of X , being a singleton, is an atom. Next we show that \mathfrak{A} is atomic, i.e every non-zero element contains a minimal non-zero element. Towards this end, let $s \in {}^\alpha \mathfrak{F}^{(0)}$ be an arbitrary sequence. Then

$$\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i \rangle_{i>1}$$

and

$$\langle \sum_{0<i<\alpha} s_i - 1, s_i \rangle_{i\geq 1}$$

are elements in y . Since

$$\{s\} = c_1\{\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i \rangle_{i>1}\} \cap c_0\{\langle \sum_{0\neq i<\alpha} s_i - 1, s_i \rangle_{i\geq 1}\},$$

it follows that $\{s\} \in A$. We have shown that A contains all singletons, hence is atomic. Call an element rectangular if $c_0x \cap c_1x = x$. As easily checked, every singleton is rectangular. Also $y = \bigcup X$, and so $y = \sup X$ exists in \mathfrak{A} . Finally, let $x = \{s\}$ be an atom of \mathfrak{A} . Then

$$c_0x \cap y = \{\langle \sum_{i>0} s_i - 1, s_i \rangle_{i>0}\}$$

and

$$c_1x \cap y = \{\langle s_0, s_0 + 1 - \sum_{i>1} s_i, s_i \rangle_{i>1}\};$$

which are singletons, hence atoms. Let

$$\tau(x, y) = c_1(c_0x \cdot s_1^0 c_1y) \cdot c_1x \cdot c_0y.$$

Let

$$Y = \{\tau(a, b) : a, b \in X\}.$$

We now show that, given that \mathfrak{A} is a neat reduct, $k = \sup Y$ exists in \mathfrak{A} . Let

$$\tau_\alpha(x, y) = c_\alpha(s_\alpha^1 c_\alpha x \cdot s_\alpha^0 c_\alpha y).$$

Then $\tau_\alpha(x, y) \leq \tau(c_\alpha x, c_\alpha y)$ and x, y are α -closed and rectangular, then $\tau_\alpha(x, y) = \tau(x, y)$. Indeed, computing we get

$$\tau_\alpha(x, y) = c_\alpha(s_\alpha^1 c_\alpha x \cap s_\alpha^0 c_\alpha y)$$

$$\leq c_\alpha(s_\alpha^1(c_0c_\alpha x \cap c_1c_\alpha x) \cap s_\alpha^0(c_0c_\alpha y \cap c_1c_\alpha y))$$

(Here, equality holds if x, y are α -closed and rectangular)

$$\begin{aligned} &= c_\alpha(s_\alpha^1c_0c_\alpha x \cap s_\alpha^1c_1c_\alpha x \cap s_\alpha^0c_0c_\alpha y \cap s_\alpha^0c_1c_\alpha y). \\ &= c_\alpha(s_\alpha^1c_0c_\alpha x \cap c_1c_\alpha x \cap c_0c_\alpha y \cap s_\alpha^0c_1c_\alpha y) \\ &= c_\alpha(s_\alpha^1c_0c_\alpha x \cap s_\alpha^0c_1c_\alpha y) \cap c_1c_\alpha x \cap c_0c_\alpha y \\ &= c_\alpha(s_\alpha^1c_0c_\alpha x \cap s_\alpha^1s_1^0c_1c_\alpha y) \cap c_1c_\alpha x \cap c_0c_\alpha y \\ &= c_\alpha s_\alpha^1(c_0c_\alpha x \cap s_1^0c_1c_\alpha y) \cap c_1c_\alpha x \cap c_0c_\alpha y \\ &= c_1s_1^\alpha(c_0c_\alpha x \cap s_1^0c_1c_\alpha y) \cap c_1c_\alpha x \cap c_0c_\alpha y \\ &= c_1s_1^\alpha c_\alpha(c_0c_\alpha x \cap s_1^0c_1c_\alpha y) \cap c_1c_\alpha x \cap c_0c_\alpha y. \\ &= c_1c_\alpha(c_0c_\alpha x \cap s_1^0c_1c_\alpha y) \cap c_1c_\alpha x \cap c_0c_\alpha y \\ &= c_1(c_0c_\alpha x \cap s_1^0c_1c_\alpha y) \cap c_1c_\alpha x \cap c_0c_\alpha y \\ &= \tau(c_\alpha x, c_\alpha y). \end{aligned}$$

Let $k = \tau_\alpha^\mathfrak{B}(y, y)$, where \mathfrak{B} is any algebra in $SC_{\alpha+1}$ such that $\mathfrak{Ad}_{SC}\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{B}$. Now $k \in A$. Let $a, b \in X$, and $i \in \{0, 1\}$. Since s_α^i is a boolean endomorphism for $i < \alpha$ we have

$$s_\alpha^0a \leq s_\alpha^0y \text{ and } s_\alpha^1b \leq s_\alpha^1y.$$

But a, b are rectangular and together with y are α -closed, we obtain that

$$\tau(a, b) = c_\alpha(s_\alpha^1a \cap s_\alpha^0b) \leq c_\alpha(s_\alpha^1y \cap s_\alpha^0y) = k.$$

This shows that $k \in B$, is an upper bound of Y . We claim that k is in fact the least such upper bound, i.e. that $k = \sup Y$. Towards this end, suppose that t is an upper bound of Y in \mathfrak{A} . We have to show that $k \leq t$. Assume, to the contrary, that this is not the case, i.e. that $k - t \neq 0$. Since \mathfrak{A} is atomic, there is an atom z say, below $k - t$ i.e.

$$z \leq k \text{ and } z \leq -t.$$

From $0 \neq z \leq w = c_\alpha(s_\alpha^1y \cap s_\alpha^0y)$, and z is α -closed, it readily follows that

$$z \cap (s_\alpha^1y \cap s_\alpha^0y) \neq 0$$

thus

$$c_0z \cap c_1z \cap (s_\alpha^1y \cap s_\alpha^0y) \neq 0.$$

From $s_\alpha^i c_i z = c_i z$, and the fact that the s_α^i 's are boolean endomorphisms for each $i < \alpha$, we get

$$s_\alpha^1(c_1z \cap y) \cap s_\alpha^0(c_0z \cap y) \neq 0.$$

Now let $a = c_1 z \cap y$. Then a is a singleton, hence $a \in At\mathfrak{C}$. Also $a \leq y$. But $y = \sup X$ and $a \cap y = a \neq 0$, therefore $a \cap x \neq 0$, for some atom $x \in X$, and so $a = x \in X$. Analogously, if $b = c_0 z \cap y$, then $b \in X$. Being atoms, a and b are rectangular, and therefore we have from

$$\tau(a, b) = c_\alpha(s_\alpha^1(c_1 z \cap y) \cap s_\alpha^0(c_0 z \cap y)).$$

and that $\tau(a, b) > 0$.

On the other hand, by invoking definitions and the fact that z is rectangular a straightforward computation yields:

$$\begin{aligned} \tau(a, b) &= \tau(c_1 z \cap y, c_0 z \cap y) \\ &= c_1(c_0(c_1 z \cap y) \cap s_1^0 c_1(c_0 z \cap y)) \cap c_1(c_1 z \cap y) \cap c_0(c_0 z \cap y). \\ &= c_1(c_0(c_1 z \cap y) \cap s_1^0 c_1(c_0 z \cap y)) \cap (c_1 z \cap c_1 y) \cap (c_0 z \cap c_0 y). \\ &= c_1(c_0(c_1 z \cap y) \cap s_1^0 c_1(c_0 z \cap y)) \cap (c_0 z \cap c_1 z) \cap (c_0 y \cap c_1 y). \\ &= c_1(c_0(c_1 z \cap y) \cap s_1^0 c_1(c_0 z \cap y)) \cap z \cap (c_0 y \cap c_1 y). \end{aligned}$$

That is, $0 < \tau(a, b) \leq z$. But z is an atom, and so $z = \tau(a, b)$. Since t is an upper bound for Y , we get $z \leq t$. But we have $z \leq -t$. Hence $z = 0$ which is impossible, since z is an atom. This means that $k \leq t$, and so $k = \sup Y$ as required. We will determine k . We start by evaluating $\tau(a, b)$ for $a, b \in X$. Towards this end, let $a, b \in X$. Assume that $a = y_r$ and $b = y_t$ with

$$r = \langle r_i : i < \alpha \rangle \text{ and } t = \langle t_i : i < \alpha \rangle$$

are elements in y .

$$\text{If } r_1 \neq t_0 \text{ or } r_i \neq t_i \text{ for some } i > 1, \text{ then } \tau(y_r, y_s) = 0.$$

Else, $r_1 = t_0$ and $r_i = t_i$ for all $i > 1$. In this case,

$$\tau(y_r, y_t) = \{s\} \text{ where } s_0 = r_0, \text{ and } s_i = t_i \text{ for all } i > 0.$$

Moreover, an easy computation shows that

$$s_0 + 2 = s_1 + 2 \sum_{i>1} s_i.$$

Now let w be the set of all such elements, i.e.

$$w = \{s \in {}^\alpha \mathfrak{F}^{(0)} : s_0 + 2 = s_1 + 2 \sum_{i>1} s_i\}.$$

Let $s \in w$. Then putting

$$r = \langle s_0, s_0 + 1 - \sum_{i>0} s_i, s_i \rangle_{i>1}$$

and

$$t = \langle s_0 + 1 - \sum_{i>0} s_i, s_1, s_i \rangle_{i>1},$$

we get $r, t \in y$ and $\tau(y_r, y_s) = \{s\}$. We have shown that

$$Y \sim \{0\} = \{\{s\} : s \in w\},$$

and so

$$w = \bigcup Y.$$

Now clearly $w \leq k$. We will show that $k \leq w$. But this easily follows from the fact that A contains all singletons. In more detail, let $z \in {}^\alpha \mathfrak{F}^{(0)} \sim w$. Then $\{z\}$, and hence ${}^\alpha \mathfrak{F}^{(0)} \sim \{z\} \in A$. Since $w \subseteq {}^\alpha \mathfrak{F}^{(0)} \sim \{z\}$, we have $\tau(a, b) \subseteq {}^\alpha \mathfrak{F}^{(0)} \sim \{z\}$, for every $a, b \in X$, i.e. ${}^\alpha \mathfrak{F}^{(0)} \sim \{z\}$ is an upper bound of Y . But this means that $k \subseteq V \sim \{z\}$, i.e. $z \notin k$. We have proved that $k \leq w$. Next we proceed to show that, in fact, $w \notin A$. (That is w is a new α dimensional element). This contradiction will show that $\mathfrak{Rd}_{SC} \mathfrak{A} \notin \mathfrak{Nr}_\alpha SC_\beta$. Let

$$Pl = \{\{s \in {}^\alpha \mathfrak{F}^{(0)} : t + \sum (r_i s_i) = 0\} : \{t, r_i : i < \alpha\} \subseteq \mathfrak{F}\}.$$

Pl consists of all hyperplanes of dimension α . For $i < \alpha$, let

$$q_i = \{s \in {}^\alpha \mathfrak{F}^{(0)} : s_i + 1 = \sum_{j \neq i} s_j\}.$$

Let $i \in \alpha$ and $k, l \in \alpha$ be distinct. Then if $i \in \{k, l\}$, $i = k$ say, then

$$\mathbf{s}_{kl} q_i = q_l.$$

Else, we have $i \notin \{k, l\}$, so that i, k, l are pairwise distinct, in which case we have

$$\mathbf{s}_{kl} q_i = q_i.$$

Now let

$$Pl^S = \{q_i : i < \alpha\}.$$

Then by the above Pl^S is closed under the operations \mathbf{s}_{kl} for all $k, l \in \alpha$. Notice that $y = q_0 \in Pl^S$. Thus Pl^S consists of all hyperplanes obtained by interchanging the k, l co-ordinates of y , for all $k, l \in \alpha$, i.e. implementing the operation \mathbf{s}_{kl} . Here the superscript S is short for substitutions (corresponding

to replacements) reflecting the fact that Pl^S consists of *substituted* versions of y . Let

$$Pl^< = \{p \in Pl : c_i p = p, \text{ for some } i < \alpha\}.$$

If $p \in Pl^<$ and $c_i p = p$, then p is a hyperplane parallel to the i -th axis.

Note that for $p \in Pl$, $p = \{s \in {}^\alpha\mathfrak{F}^{(0)} : t + \sum_i r_i s_i = 0\}$ say, then $c_i p = p$ (i.e. p is parallel to the i -th axis) iff $r_i = 0$. Note too, that if $p \in Pl^<$ and $k, l \in \alpha$ then $s_{kl} p \in Pl^<$.

In the following we summarize the above, and state some other facts easy to check as well.

- (1) $y = q_0 \in Pl^S$.
- (2) $Pl^S \cup \{w, d_{ij} : i, j \in \alpha\} \subseteq Pl$.
- (3) If $q \in Pl^S(Pl^<)$ and $i, j \in \alpha$, then $s_{ij} q \in Pl^S(Pl^<)$.
- (4) $Pl^S \cap Pl^< = \emptyset, w \notin Pl^<$ and ${}^\alpha\mathfrak{F}^{(0)} \in Pl^<$.
- (5) $\{d_{ij} : i \neq j, i, j \in \alpha\} \subseteq Pl^<$ iff $\alpha \geq 3$.

For further use, we shall need:

Notation .

- (i) Let X be a set. Then $Z \subseteq_\omega X$ abbreviates “ Z is a finite subset X ”.
 $\wp_\omega X$ denotes all finite subsets of X , i.e.

$$\wp_\omega X = \{Z \in \wp(X) : Z \subseteq_\omega X\}.$$

- (ii) Let $\mathfrak{A} \in K_\alpha$, $a \in A$ and $\Delta \subseteq_\omega \alpha$. Then $c_{(\Delta)} a = c_{i_0} \cdots c_{i_{n-1}} a$, where $\Delta = \{i_0, \dots, i_{n-1}\}$. Because cylindrifications commute, the definition of $c_{(\Delta)}$ does not depend on any linear order defined on Δ . When $n = 0$, i.e. Δ is empty, then $c_{(\Delta)} a = c_\emptyset a := a$.
- (iii) To simplify notation, from now on we may write $-a$ instead of ${}^\alpha\mathfrak{F}^{(0)} \sim a$. It will be clear from context whether $-$ refers to the operation of subtraction in \mathfrak{F} or the operation of complementation in \mathfrak{A} . We also write $1^\mathfrak{A}$, or just 1, instead of the more cumbersome ${}^\alpha\mathfrak{F}^{(0)}$.

Some more definitions.

$$G = \{q, -q, p, -p, c_{(\Delta)}\{0\}, -c_{(\Delta)}\{0\} : q \in Pl^S, p \in Pl^< \cup \{d_{01}\}, \Delta \subseteq_\omega \alpha, 0 \in \Delta\}.$$

Forming finite intersections of elements in G , we let

$$G^* = \{\bigcap_{i \in n} g_i : n \in \omega, g_i \in G\},$$

and forming finite unions of elements in G^* , we let

$$G^{**} = \left\{ \bigcup_{i \in n} g_i : n \in \omega, g_i \in G^* \right\}.$$

It is easy to see that $\{y, y_s : s \in y\} \subseteq G^{**}$, and G^{**} is a boolean field of sets. We start by proving that $w \notin G^{**}$. To this end, we set:

$$L = \{p \in Pl^< : c_0 p \neq p\} \text{ and } P(0) = L \cup \{d_{01}\}.$$

Notice that $L = P(0)$ iff $\alpha > 2$. L stands for the set of “lines” not parallel to the 0th axis. Being in $Pl^<$, any such line is parallel to some other axis. When $\alpha = 2$, i.e. in the plane $\mathfrak{F} \times \mathfrak{F}$, these are precisely the lines that are parallel to the “ y axis”. Next we define

$$G_1 = \{g \in G^* : g \subseteq q, \text{ for some } q \in Pl^S\}$$

and

$$G_2 = \{g \in G^* : g \not\subseteq q, \text{ for all } q \in Pl^S \text{ and } g \subseteq p, \text{ for some } p \in P(0)\}.$$

Note that $Pl^S \subseteq G_1$ and that $P(0) \subseteq G_2$. Note too, that $G_1 \cap G_2 = \emptyset$. Now let

$$G_3 = \{p_1 \cap p_2 \dots \cap p_k : k \in \omega, \{p_1, p_2, \dots, p_k\} \subseteq G \sim (Pl^S \cup P(0))\}.$$

It is easy to see that $G^* = G_1 \cup G_2 \cup G_3$. To prove that $w \notin G^{**}$ we need:

Claim . *If $g \in G_3$ and $0 \neq g$, then $g \not\subseteq w$.*

Proof of Claim

Assume that $g = p_1 \cap p_2 \dots \cap p_k$ say, with $p_i \in G$ and $p_i \notin (Pl^S \cup P(0))$ for $1 \leq i \leq k$, and let $z \in g$. Then for $1 \leq i \leq k$, we have in addition that

$$p_i \in \{p, -p, c_{(\Delta)}\{0\}, -c_{(\Delta)}\{0\}, -q : p \in Pl^< \cup \{d_{01}\}, q \in Pl^S, \Delta \subseteq_\omega \alpha, 0 \in \Delta\}.$$

Let $[]$ be the function from G into $\wp(\mathfrak{F})$ defined as follows:

$$[p] = \{1/r_0(-t - \sum_{0 \neq i < \alpha} r_i z_i)\} \text{ if } p = -\{s \in^\alpha \mathfrak{F}^{(0)} : t + \sum_{i < \alpha} r_i s_i = 0\}, r_0 \neq 0.$$

Else

$$[p] = 0.$$

Let

$$r \in \mathfrak{F} \sim ((\bigcup_{1 \leq i \leq k} [p_i]) \cup [-w])$$

be arbitrary, and let

$$z_r^0 = z \sim \{(0, z_0)\} \cup \{(0, r)\}.$$

By the choice of $r \notin [-w]$ we have

$$r \neq -2 + z_1 + 2 \sum_{i>1} z_i.$$

Hence $z_r^0 \notin w$. If $p_i = -q$ with $q \in Pl^S$, then similarly, $r \notin [-q]$ hence $z_r^0 \notin p_i$. Now assume that $p_i \in Pl^<$. Then we have $p_i \notin P(0)$ hence $c_0 p_i = p_i$. Since $z \in p_i$ it follows that $z_r^0 \in c_0 p_i$. But $p_i \notin P(0)$ hence $z_r^0 \in c_0 p_i = p_i$. If $p_i = -p$ where $p \in Pl^<$, then $z \notin p$ and so $z_r^0 \notin p$ since again $c_0 p = p$. That is if $z_r^0 \in p$ then z differing from z_r^0 in atmost the 0th place would also be in p . Now assume that $p_i = c_{(\Delta)}\{\mathbf{0}\}$. Then if $z \in p_i$ then $z_r^0 \in p_i$ since $c_0 p_i = p_i$ by $0 \in \Delta$. Finally if $p_i = -c_{(\Delta)}\{\mathbf{0}\}$ then $z \notin c_{(\Delta)}\{\mathbf{0}\}$ iff $z_r^0 \notin c_{(\Delta)}\{\mathbf{0}\}$ by $0 \in \Delta$. We have shown that

$$z_r^0 \in g \sim w \text{ i.e. } g \not\subseteq w.$$

We now proceed to show that $w \notin G^{**}$. Assume to the contrary, that

$$w = \bigcup \{g_i^1 : i < n_1\} \cup \bigcup \{g_i^2 : i < n_2\} \cup \bigcup \{g_i^3 : i < n_3\}$$

where

$$\{g_i^j : i < n_j\} \subseteq G_j \text{ for } j \in \{1, 2, 3\}.$$

By the above, we have $g_i^3 = 0$, for all $i < n_3$. From the definition of G_1 and G_2 , it follows that

$$w \subseteq \bigcup \{q_j : j < n_1\} \cup \bigcup \{p_j : j < n_2\}$$

where

$$\{q_j : j < n_1\} \subseteq Pl^S$$

and

$$\{p_j : j < n_2\} \subseteq P(0).$$

Since $w \subseteq -d_{01}$ for $\alpha = 2$, and $P(0) = L$ for $\alpha > 2$ we can, and will, assume that

$$\{p_j : j < n_2\} \subseteq L.$$

Before proceeding, let us take a special case as an illustration. Assume that $\alpha = 3$. Then each p_j is a hyperplane that is determined by an equation. Assume that the following equations determine the planes p_j $j < n_2$.

$$(1) \quad t_0 + r_{00}x_0 + r_{01}x_1 + r_{02}x_2 = 0$$

$$t_1 + r_{10}x_0 + r_{11}x_1 + r_{12}x_2 = 0$$

...

...

$$t_{n_2-1} + r_{n_2-1,0}x_0 + r_{n_2-1,2}x_2 = 0.$$

In each of these equation one of the coefficients other than the zeroth is equal to 0. The zeroth coefficient is *not* zero. Therefore, these equations determine hyperplanes parallel to one of the axis other than the zeroth axis. Now consider the equations

$$(2) \quad x_0 + 2 = x_1 + 2x_2$$

and

$$(3) \quad x_{\pi(0)} + 1 = x_{\pi(1)} + x_{\pi(2)}$$

where π is a permutation of $\{0, 1, 2\}$. (3) consists of 6 equations, only three of which are distinct (because of commutativity). Such equations represent the q_i 's. Then it can be easily seen that there is an s that satisfies (2) but does not satisfy (the equations in) (1) and (3).

And indeed in the general case, it can be seen by implementing easy linear algebraic arguments that, for every $n \in \omega$, for every $m \in \omega \cap (\alpha + 1)$, and for every system

$$t_0 + \sum (r_{0i}x_i) = 0$$

...

...

$$t_n + \sum (r_{ni}x_i) = 0,$$

of equations, such that for all $j \leq n$, there exists $i < \alpha$, such that

$$r_{ji} = 0 \text{ and } r_{j0} \neq 0,$$

the equation

$$x_0 + 2 = x_1 + 2 \sum x_j$$

has a solution s in the weak space ${}^\alpha\mathfrak{F}^{(0)}$, such that $s \notin q_i$ for every $i < m$, and such that s is not a solution of

$$t_j + \sum (r_{ji}x_i) = 0,$$

for every $j \leq n$. Geometrically (and intuitively) a finite union of hyperplanes cannot cover a hyperplane, unless it is one of them. By taking $n = n_1$ and $m = n_2$ we get the desired conclusion. Let us prove this formally. It clearly suffices to show that for all positive integers m, n the equation

$$x_0 + 2 = x_1 + 2 \sum_{1 \leq i \leq m} x_i$$

has a solution which is not a solution of any of the following equations:

$$(4) \quad t_k + \sum_{j \leq m} r_{kj} x_j = 0,$$

$$(5) \quad x_l + 1 = \sum_{1 \neq i \leq m} x_i,$$

where $k = 1, \dots, n$, each $r_{k0} \neq 0$, for each $k = 1, \dots, n$ there is a j with $0 < j \leq m$ such that $r_{kj} = 0$, and $0 \leq l \leq m$. To do this, first substitute $x_0 = -2 + x_1 + 2 \sum_{1 < i \leq m} x_i$ into each of the equations (4) and (5), obtaining

$$(6) \quad t_k - 2r_{k0} + (r_{k0} + r_{k1})x_1 + \sum_{1 < j \leq m} (2r_{k0} + r_{kj})x_j = 0,$$

$$(7) \quad -1 + \sum_{1 < j \leq m} x_j = 0,$$

$$(8) \quad 3 + \sum_{1 < j \leq m} (-3x_j) = 0,$$

$$(9) \quad 3 - 2x_1 - x_k + \sum_{j \leq m, j \notin \{0,1,k\}} (-3x_j) = 0 \text{ for } k > 1.$$

Now we define s_1, \dots, s_m by recursion. Choose s_1 so that

$$t_k - 2r_{k0} + (r_{k0} + r_{k1})s_1 \neq 0$$

for each k such that $r_{k0} + r_{k1} \neq 0$. Having defined s_t with $t < m$ choose s_{t+1} so that

$$t_k - 2r_{k0} + (r_{k0} + r_{k1})s_1 + \sum_{1 < j \leq t+1} (2r_{k0} + r_{kj})s_j \neq 0$$

for each k for which $2r_{k0} + r_{k,t+1} \neq 0$; also, if $t+1 = m$, assure that the equations (7) – (9) all fail. Finally let $s_0 = -2 + s_1 + 2 \sum_{1 < i} s_i$. Clearly the desired conclusion holds. We have proved that $w \notin G^{**}$.

To show that $w \notin A$, we will show that G^{**} is closed under the polyadic set operations. It only remains to show that G^{**} is closed under cylindrifications and substitutions, since by definition, it is a boolean field of sets and contains the diagonal elements. (Recall that for $\alpha > 2$, $\mathbf{d}_{ij} \in Pl^{<}$.)

(1) G^{**} is closed under cylindrifications.

It is enough to show that (since the \mathbf{c}_i 's are additive), that for $j \in \alpha$ and $g \in G^*$ arbitrary, we have $\mathbf{c}_j g \in G^{**}$. For this purpose, put for every $p \in Pl$

$$p(j|0) = \mathbf{c}_j \{s \in p : s_j = 0\} \text{ and } (-p)(j|0) = -p(j|0).$$

Then it is not hard to see that

$$p(j|0) = \{s \in {}^\alpha\mathfrak{F}^{(0)} : t + \sum_{i \neq j} (r_i s_i) = 0\},$$

if

$$p = \{s \in {}^\alpha\mathfrak{F}^{(0)} : t + \sum_{i < \alpha} (r_i s_i) = 0\},$$

Indeed assume that $s_1 \in \mathbf{c}_j\{s \in p : s_j = 0\}$. Then there exists s_2 in p such that s_1 and s_2 agree at all components except the j th where $s_2(j) = 0$. Then

$$0 = t + \sum_{i \neq j} r_i s_1(i) = t + \sum_{i \neq j} r_i s_2(i).$$

The other inclusion is analogous. Assume that $s_1 \in p$ and $r_j \neq 0$ such that $t + \sum_{i \neq j} r_i s_1(i) = 0$. Then $s_1(j) = 0$. Hence $s_1 \in \mathbf{c}_j\{s \in P : s_j = 0\}$. It follows thus that

$$p(j|0) \in Pl^< \text{ for every } p \in Pl.$$

Now let j and g be as indicated above. We can assume that

$$g = q_1 \cdots \cap q_l \cap p_1 \cap \cdots \cap p_n \cap -Q_1 \cdots \cap -Q_L \\ \cap -P_1 \cdots \cap -P_m \cap y \cap -\mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -\mathbf{c}_{(\Delta_N)}\{\mathbf{0}\},$$

where

$$l, L, n, m, N \in \omega, q_i, Q_i \in Pl^S, p_i, P_i \in Pl^< \cup \{\mathbf{d}_{01}\}, \\ \mathbf{c}_j p_i \neq p_i, \mathbf{c}_j P_i \neq P_i, \mathbf{c}_j q_i \neq q_i, \mathbf{c}_j Q_i \neq Q_i, \\ y \in \{\mathbf{c}_{(\Delta)}\{\mathbf{0}\}, 1 : \Delta \in \wp_\omega \alpha, 0 \in \Delta, j \notin \Delta\},$$

and

$$\{\Delta_1, \dots, \Delta_n\} \subseteq \{x \in \wp_\omega \alpha : j \notin x, 0 \in x\}.$$

This is so because

$$\mathbf{c}_j(x \cap \mathbf{c}_j y) = \mathbf{c}_j x \cap \mathbf{c}_j y, \\ \text{if } j \notin \Delta \text{ then } \mathbf{c}_j x = x$$

and

$$\mathbf{c}_{(\Delta)}\{\mathbf{0}\} \cap \mathbf{c}_{(\Gamma)}\{\mathbf{0}\} = \mathbf{c}_{(\Delta \cap \Gamma)}\{\mathbf{0}\}.$$

We distinguish between 2 cases:

Case 1.

$$y = \mathbf{c}_{(\Delta)}\{\mathbf{0}\} \text{ and } j \notin \Delta.$$

Then a lengthy but routine computation gives

$$\begin{aligned}
& \mathbf{c}_j(q_1 \cdots \cap q_l \cap -Q_1 \cdots \cap -Q_L \cap p_1 \cdots \cap p_n \cap -P_1 \cdots \cap -P_m \\
& \quad \cap \mathbf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -\mathbf{c}_{(\Delta_N)}\{\mathbf{0}\}) \\
& = q_1(j|0) \cap q_l(j|0) \cap -Q_1(j|0) \cdots \cap -Q_L(j|0) \cap p_1(j|0) \cdots \\
& \quad \cap p_n(j|0) \cap -P_1(j|0) \cdots \cap -P_m(j|0) \\
& \quad \cap \mathbf{c}_j \mathbf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathbf{c}_j \mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -\mathbf{c}_j \mathbf{c}_{(\Delta_N)}\{\mathbf{0}\}.
\end{aligned}$$

Indeed let

$$\begin{aligned}
s & \in \mathbf{c}_j(q_1 \cdots \cap q_l \cap -Q_1 \cdots \cap -Q_L \cap p_1 \cdots \cap p_n \cap -P_1 \cdots \cap -P_m \\
& \quad \cap \mathbf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -\mathbf{c}_{(\Delta_N)}\{\mathbf{0}\}).
\end{aligned}$$

For a sequence t , we write t_u^j for the sequence that agrees with t except at j where $t_u^j(j) = u$. Now, by definition of cylindrifications, there exists $u \in \mathfrak{F}$ such that

$$\begin{aligned}
s_u^j & \in (q_1 \cdots \cap q_l \cap -Q_1 \cdots \cap -Q_L \cap p_1 \cdots \cap p_n \cap -P_1 \cdots \cap -P_m \\
& \quad \cap \mathbf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -\mathbf{c}_{(\Delta_N)}\{\mathbf{0}\}).
\end{aligned}$$

Now since $s_u^j \in q_1$ and $s_u^j \in \mathbf{c}_{(\Delta)}\{\mathbf{0}\}$ and $j \notin \Delta$ it readily follows that $u = 0$. But then $s \in q_1(j|0)$. Similarly $s \in q_i(j|0)$ for every $i \leq l$ and $s \in p_i(j|0)$ for each $i \leq n$. It is clear that $s \in \mathbf{c}_j \mathbf{c}_{(\Delta)}\{\mathbf{0}\}$. Also $s \notin Q_i(j|0)$ for any $i \leq L$ for else $s_0^j \in Q_i$ which is not the case. Same reasoning gives $s \notin P_i(j|0)$ for all $i \leq m$. Finally $s \notin \mathbf{c}_j \mathbf{c}_{(\Delta_i)}\{\mathbf{0}\}$ for $i \leq N$, because $s_u^j \notin \mathbf{c}_{(\Delta_j)}\{\mathbf{0}\}$. Now conversely if we start with

$$\begin{aligned}
s & \in q_1(j|0) \cap q_l(j|0) \cap -Q_1(j|0) \cdots \cap -Q_L(j|0) \cap p_1(j|0) \cdots \\
& \quad \cap p_n(j|0) \cap -P_1(j|0) \cdots \cap -P_m(j|0) \\
& \quad \cap \mathbf{c}_j \mathbf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathbf{c}_j \mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -\mathbf{c}_j \mathbf{c}_{(\Delta_N)}\{\mathbf{0}\}.
\end{aligned}$$

Then

$$\begin{aligned}
s_0^j & \in (q_1 \cdots \cap q_l \cap -Q_1 \cdots \cap -Q_L \cap p_1 \cdots \cap p_n \cap -P_1 \cdots \cap -P_m \\
& \quad \cap \mathbf{c}_{(\Delta)}\{\mathbf{0}\} \cap -\mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap -\mathbf{c}_{(\Delta_N)}\{\mathbf{0}\}).
\end{aligned}$$

Case 2.

$$y = 1$$

This case is harder, so we start with some special cases before embarking on the most general case. Consider the case when

$$g = q \cap p \cap -P \cap -c_{(\Delta)}\{\mathbf{0}\},$$

where $q \in Pl^S$, and $p, P \in Pl^<$. Then

$$\begin{aligned} c_j g &= c_j(q \cap p) \cap c_j(q - P) \cap c_j(q - c_{(\Delta)}\{\mathbf{0}\}) \\ &\quad \cap c_j(p - P) \cap c_j(p - c_{(\Delta)}\{\mathbf{0}\}). \end{aligned}$$

Let us prove this special case to visualise matters. The general case will be a little bit more involved, but all the same an immediate generalization. Let

$$\begin{aligned} s &\in c_j(q \cap p) \cap c_j(q - P) \cap c_j(q - c_{(\Gamma)}\{\mathbf{0}\}) \\ &\quad \cap c_j(p - P) \cap c_j(p - c_{(\Delta)}\{\mathbf{0}\}). \end{aligned}$$

Then there exist $u_1, u_2, u_3, u_4, u_5 \in \mathfrak{F}$ such that

$$\begin{aligned} s_{u_1}^j &\in q \cap p, s_{u_2}^j \in q - P, s_{u_3}^j \in q - c_{(\Gamma)}\{\mathbf{0}\}, \\ s_{u_4}^j &\in p - P \text{ and } s_{u_5}^j \in p - c_{(\Delta)}\{\mathbf{0}\}. \end{aligned}$$

Now $c_j p \neq p$ it follows that $u_4 = u_5$. Similarly u_1 and u_2 and u_3 are also equal because $c_j q \neq q$. But note that $s_{u_1}^j \in p \cap q$ and $c_j q \neq q$, it follows that all of the u_i 's are in fact equal to u say. It readily follows that

$$s_u^j \in q \cap p \cap -P \cap -c_{(\Delta)}\{\mathbf{0}\},$$

hence

$$s \in c_j(q \cap p \cap -P \cap -c_{(\Delta)}\{\mathbf{0}\}).$$

The other inclusion is much easier, in fact it is absolutely straightforward. Start with

$$s \in c_j(q \cap p \cap -P \cap -c_{(\Delta)}\{\mathbf{0}\}).$$

Then

$$s_u^j \in (q \cap p \cap -P \cap -c_{(\Delta)}\{\mathbf{0}\}).$$

It follows that

$$\begin{aligned} s_u^j &\in q \cap p, s_u^j \in q - P, s_u^j \in q - c_{(\Gamma)}\{\mathbf{0}\}, \\ s_u^j &\in p - P \text{ and } s_u^j \in p - c_{(\Delta)}\{\mathbf{0}\}. \end{aligned}$$

The required follows. Now assume (still considering a special case) that

$$g = q \cap p_1 \cap \cdots \cap p_n \cap \cdots \cap -P_1 \cdots \cap -P_m$$

$$\cap - \mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap - \mathbf{c}_{(\Delta_N)}\{\mathbf{0}\},$$

then by the same token

$$\mathbf{c}_j g = \cap_{k \leq n} \mathbf{c}_j(q \cap p_k) \cap \cap_{i \leq m} \mathbf{c}_j(q - P_i) \cap_{i \leq N} \mathbf{c}_j(q - \mathbf{c}_{\Delta_N}\{\mathbf{0}\})$$

$$\cap_{k \leq n} ((\cap_{i \leq n} \mathbf{c}_j(p_k \cap p_i) \cap_{i \leq m} \mathbf{c}_j(p_k - P_i) \cap_{i \leq N} \mathbf{c}_j(p_k - \mathbf{c}_{(\Gamma)}\{\mathbf{0}\})).$$

To illustrate matters further, paving the way for the general case, consider another special case that is essentially different than the one just considered.

$$g = -Q \cap p_1 \cdots p_n \cap -P_1 \cdots \cap -P_m$$

$$\cap - \mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap - \mathbf{c}_{(\Delta_N)}\{\mathbf{0}\}.$$

Then

$$\mathbf{c}_j g = \cap_{k \leq n} \mathbf{c}_j(p_k - Q)$$

$$\cap_{k \leq n} ((\cap_{i \leq n} \mathbf{c}_j(p_k \cap p_i) \cap_{i \leq m} \mathbf{c}_j(p_k - P_i) \cap_{i \leq N} \mathbf{c}_j(p_k - \mathbf{c}_{(\Delta_i)}\{\mathbf{0}\})).$$

Now consider the general case. We assume for better readability that

$$g = p_1 \cap \cdots \cap p_n \cap \cdots \cap -P_1 \cdots \cap -P_m$$

$$\cap - \mathbf{c}_{(\Delta_1)}\{\mathbf{0}\} \cdots \cap - \mathbf{c}_{(\Delta_N)}\{\mathbf{0}\},$$

where $p_i, P_i \in Pl^< \cup Pl^S$. Here we have collected the planes in $Pl^<$ and Pl^S together to simplify matters. Now we have by the same reasoning as above

$$\mathbf{c}_j g = \cap_{k \leq n} ((\cap_{i \leq n} \mathbf{c}_j(p_k \cap p_i) \cap \cap_{i \leq m} \mathbf{c}_j(p_k - P_i)$$

$$\cap_{i \leq N} \mathbf{c}_j(p_k - \mathbf{c}_{(\Delta_i)}\{\mathbf{0}\})).$$

Now for every $p, q \in Pl$, there are p', q', p'' and $q'' \in Pl^<$ such that (*)

$$\mathbf{c}_j(p \cap q) = p' \cap q',$$

$$\mathbf{c}_j(p \sim q) = p'' \sim q''$$

and if $j \in \Delta p \sim \Gamma$, then (**)

$$\mathbf{c}_j(p \sim \mathbf{c}_{(\Gamma)}\{\mathbf{0}\}) = {}^\alpha \mathfrak{F}^{(0)} \sim p(j|0) \cup (p(j|0) \sim \mathbf{c}_j \mathbf{c}_{(\Gamma)}\{\mathbf{0}\}).$$

Let us consider (*). We give an illustration when $\alpha = 2$ and $\mathfrak{F} = \mathfrak{R}$. In this case elements in $Pl^<$ are either the whole plane or horizontal lines or vertical lines. If p and q are any two straight lines that are not parallel, then they intersect at a point. If we cylindrify (this point) with respect to one of the co-ordinates we get a line that is indeed parallel to one of the axis. Next, if we have two straight lines p and q , say, that intersect

and neither is parallel to one of the axis, then $p - q$ will be p “minus” one point (the point of intersection). If we cylindrify with respect to one of the co-ordinates then we get the plane minus the straight line that is parallel to one of the axis; call this straight line p' . But this is nothing more than $1 - p'$. On the other hand, if p is parallel to one of the axis and we cylindrify p in the direction of this axis, we get $p - q$ where q is the line orthogonal to p and passes by the missing point, hence q is parallel to the other axis. If p and q are parallel then $p - q = p$ and so for $j \in \{0, 1\}$, we have $c_j(p - q) = c_j p \in Pl^<$.

Now we turn to $(**)$. Assume that $p = \{s \in {}^0\mathfrak{F}^{(0)} : t + \sum_i r_i s_i = 0\}$. Let $j \in \Delta p \sim \Gamma$. Let $s \in c_j(p - c_{(\Gamma)}\{0\})$. Then there is some $u \in \mathfrak{F}$ such that $s_u^j \in p$ and $s_u^j \notin c_{(\Gamma)}\{0\}$. Since $j \in \Delta p$ we have $c_j p \neq p$ hence $r_j \neq 0$. Let $u = 1/r_j(t + \sum_{i \neq j} r_i s_i)$. Then we have two cases. If $u \neq 0$, then $s \notin p(j|0)$ for else there exists $q \in p$ such that q differs from s in at most the j th component and $q(j) = 0$. But $q(j) = -1/r_0(t + \sum_{i \neq j} r_i s_i) = u$. This would imply that $u = 0$ which is a contradiction. Else $u = 0$. Then $s \in p(j|0)$ because $s_u^j \in p$ and $s_u^j(j) = 0$. Also $s \notin c_j c_{(\Gamma)}\{0\}$ for else $s_u^j \in c_{(\Gamma)}\{0\}$. We leave the other inclusion to the reader. We have proved that $c_j g \in G^{**}$.

(2) G^{**} is closed under substitutions.

It is enough to consider the s_{ij} 's, since for all $i, j \in \alpha$ $s_i^j(x) = c_i(x \cap d_{ij})$ i.e. the s_i^j 's are term definable, G^{**} is closed under the c_i 's and $d_{ij} \in G^{**}$. For this purpose, let

$$H = Pl^S \cup Pl^< \cup \{c_{(\Delta)}\{0\} : \Delta \subseteq_\omega \alpha\}.$$

Here we are not requiring that $0 \in \Delta$, so that H is bigger than G . However, since G^{**} is closed under cylindrifications we still have $G \subseteq H \subseteq G^{**}$. Now let $i, j \in \alpha$, and $x \in H$. Then it not hard to check that $s_{ij}x \in G^{**}$. Indeed Pl^S is closed under s_{ij} and so is $Pl^<$. Assume that $y = c_{(\Delta)}\{0\}$, $\Delta \subseteq_\omega \alpha$, and $i, j < \alpha$ are distinct. If $\{i, j\} \subseteq \Delta$ or $\{i, j\} \cap \Delta = \emptyset$, then $s_{ij}c_{(\Delta)}\{0\} = c_{(\Delta)}\{0\}$. Now suppose that $i \in \Delta$, $j \notin \Delta$ and that $c_{(\Delta)}\{0\} = V_1 \times V_2 \times V_l \dots$ where $V_l \in \{\mathfrak{F}, \{0\}\}$ and $V_l = \mathfrak{F}$ only for finitely many l 's. Assume further that $i < j$, $V_i = \mathfrak{F}$ and $V_j = \{0\}$ so that $c_{(\Delta)}\{0\} = V_1 \times \dots \mathfrak{F} \times \dots \{0\} \times \dots$. Then $s_{ij}c_{(\Delta)}\{0\}$ is obtained from $c_{(\Delta)}\{0\}$ by interchanging the i th and j th co-ordinates, that is it is equal to $V_1 \times \dots \{0\} \times \dots \mathfrak{F} \times \dots$, which is of the form $c_{(\Gamma)}\{0\}$ for some $\Gamma \subseteq_\omega \alpha$. In fact $\Gamma = (\Delta \sim \{i\}) \cup \{j\}$. By noting that the s_{ij} 's are boolean endomorphisms we get that G^{**} , generated as a boolean algebra by H (since $G \subseteq H$ is a set of generators), is closed under s_{ij} .

It thus follows that $w \notin A$ and we are done. ■

4 The class of weakly neat atom structures is not elementary

From the title, it is clear that there exists atom structures such that are very weakly neat.

Theorem 4.1. *The class of weakly neat atom structures is not elementary, for finite and infinite dimensions.*

Proof. For the first part there is an atomic $\mathfrak{A} \in \mathfrak{Nr}_n \mathbf{CA}_\omega$ and an atomic \mathfrak{B} that is not a neat reduct, elementary equivalent to \mathfrak{A} . $\text{At}\mathfrak{A}$ is strongly neat, but $\text{At}\mathfrak{B}$ is not. But $\text{At}\mathfrak{A} \equiv \text{At}\mathfrak{B}$, because atom structures are first order interpretable in the algebras, and we are done. \square

The second part is harder. The construction is similar, but not identical to the one used in [20]. We prove our result for the harder infinite dimensional case. The finite dimensional case can be easily distilled from the finite version of our algebras constructed in [20]. Throughout this section α is a fixed arbitrary ordinal with $\alpha \geq \omega$ and R is an uncountable set with $|R| > |\alpha|$.

Notation . Let s be an α -ary sequence, and $i < \alpha$. Then $s(i|w)$ denotes the α -ary sequence for which: $s(i|w)(i) = w$ and $s(i|w)(x) = s(x)$ for $x \neq i$. For a set U , $w \in W$ and β an ordinal, we let \mathbf{w} denote the β -ary sequence $\beta \times \{w\}$, i.e the constant sequence for which $\mathbf{w}_i = w$, for all $i \in \beta$. The arity of \mathbf{w} , we hope, will be clear from context.

Lemma 4.2. *There are a set U , $w \in U$, and α -ary relations $C_r \subseteq {}^\alpha U^{(\mathbf{w})}$ for all $r \in R$, such that conditions (i) and (ii) below hold:*

(i) *For all $r \in R$, for all $i < \alpha$, for all $s \in {}^\alpha U^{(\mathbf{w})}$, there exists $y \in U - Rgs$ such that $s(i|y) \in C_r$.*

(ii) *The C_r 's ($r \in R$) are pairwise disjoint.*

Notation . Let $\mathbf{U} = \langle U, C_r \rangle_{r \in R}$, with $C_r \subseteq {}^\alpha U^{(\mathbf{w})}$, be the structure described in Lemma 1*. Let $W = \cup_{i \in \alpha} U \times \{i\}$. Then W is simply $|\alpha|$ disjoint copies of U . Let $s = \langle (w, 0), \dots, (w, j) \dots \rangle_{j < \alpha}$. Then $s \in {}^\alpha W$ with $s(i) \in W_i = U \times \{i\}$. (Recall that $\mathbf{w} = \langle w : i < \alpha \rangle$). Let V be the weak space ${}^\alpha W^{(s)}$ and let $C(\alpha) = \langle Sb(V), \cup, \cap, c_i, d_{ij} \rangle$ be the full Ws_α with unit V . S_α denotes the set of one to one functions (equivalently permutations) which are almost everywhere identity, i.e, which are in $\alpha^\alpha^{(Id)}$.

Let $u = \langle u_0, u_1, \dots, u_j \dots \rangle_{j \in \alpha} \in S_\alpha$, and $r \in R$. Then $p(u, r)$ denotes the following α -ary relation on V :

$$\{ \langle (a_0, u_0), \dots, (a_j, u_j) \dots \rangle_{j < \alpha} \in V : \langle a_0, \dots, a_j, \dots \rangle_{j < \alpha} \in C_r \}.$$

$$P(u) = \{p(u, r) : r \in R\}.$$

Note that $P(u)$ consists of $|R|$ many α relations on V . Let N be a fixed countably infinite subset of R . Then $P_\omega(u) = \{p(u, r) : r \in N\}$. Again $P_\omega(u)$ countably infinite. Now we define $\mathfrak{A}(\alpha) \in W_{S_\alpha}$ and $\mathfrak{B}(\alpha) \subseteq \mathfrak{A}(\alpha)$ as follows:

$$\mathfrak{A}(\alpha) = \mathfrak{S}g^{C(\alpha)}(\cup\{P(u) : u \in S_\alpha\})$$

$$\mathfrak{B}(\alpha) = \mathfrak{S}g^{A(\alpha)}((\cup\{P(u) : u \in S_\alpha - \{Id\}\}) \cup P_\omega(Id)).$$

Theorem 4.3. $\mathfrak{A}(\alpha)$ and $\mathfrak{B}(\alpha)$ are atomic algebras with elementary equivalent atom structures. $\text{At}\mathfrak{B}$ is not weakly neat, but $\text{At}\mathfrak{A}$ is.

Proof. The thing to notice here, is that $\mathfrak{A}(\alpha)$ is a disjoint union of algebras whose boolean part are isomorphic to the finite-cofinite Boolean algebra on an uncountable set R . We have ${}^\alpha\alpha^{(Id)}$ many copies. Hence it is atomic. Also \mathfrak{B} is atomic. We will show that that $\mathfrak{B} \equiv \mathfrak{A}$, hence $\text{At}\mathfrak{A} \equiv \text{At}\mathfrak{B}$, and that any algebra having an atom structure isomorphic to $\text{At}\mathfrak{B}$ is not a neat reduct. \square

We shall start by showing that $\mathfrak{A}(\alpha)$ and $\mathfrak{B}(\alpha)$ are elementary equivalent, so that their atom structures will also be elementary equivalent, and that $\mathfrak{B}(\alpha)$ is not a neat reduct; not only that, but any algebra based on its atom structure is not a neat reduct.

Then we shall impose further restrictions on the structure $\langle U, C_r \rangle_{r \in R}$, that will force $A(\alpha)$ to be a neat reduct (and not just a subneat reduct).

Notation . Let $u = \langle u_0, u_1, \dots, u_j, \dots \rangle_{j \in \alpha} \in {}^\alpha\alpha^{(Id)}$. Then 1_u denote the α -ary relation on U :

$$\{\langle (a_0, u_0), (a_1, u_1), \dots, (a_j, u_j), \dots \rangle_{j < \alpha} \in V : \langle a_0, \dots, a_j, \dots \rangle_{j \in \alpha} \in {}^\alpha U^{(\mathbf{w})}\}.$$

L stands for 1_{Id} .

Theorem 4.4. (i) For all $u \in S_\alpha$, $1_u \in A(\alpha)$.

(ii) Let $Rl_L A(\alpha) = \{x \in A(\alpha) : x \leq L\}$. Then $Rl_L A(\alpha)$ is atomic and its set of atoms is equal to $P(Id)$. Furthermore, $Rl_L A(\alpha)$ is generated as a BA by $P(Id)$.

(iii) For all non zero $a \in Rl_L A(\alpha)$, for all $i < \alpha$, $c_i a = c_i L$.

(iv) For all $a \in A(\alpha)$, for all $i < \alpha$, $c_i a \cap L \in \{0, L\}$.

(v) Let $P = \cup\{P(u) : u \in S_\alpha\}$. Let p be a permutation of $P(Id)$. Then there exists an automorphism f of $A(\alpha)$, such that $p \subseteq f$ and $f(x) = x$, for each $x \in P - P(Id)$.

Proof.

- (i) Let $u \in S_\alpha$. Let $r_1, r_2 \in R$. Then as easily checked $c_0(p(u, r_1)) \cap c_1(p(u, r_2)) = 1_u$.
- (ii) Since the C_r 's are pairwise disjoint, we have $p(Id, r_1) \cap p(Id, r_2) = 0$ for distinct r_1 and r_2 . Thus the set of atoms of $Rl_L A(\alpha)$ is precisely $P(Id)$. By $c_i(p(Id, r)) \cap 1_{Id} = 1_{Id}$ and $p(Id, r) \cap d_{ij} = 0$, for distinct $i, j < \alpha$, it follows that $Rl_L A(\alpha)$ is indeed generated as a BA by the set $P(Id)$.
- (iii) Let $a \in Rl_L A(\alpha)$ be non zero. By (ii) there exists an atom $p(Id, r)$ below a . Let $i < \alpha$. Then $c_i(p(Id, r)) = (\text{by Lemma 1 (i)}) \cup \{1_u : u \in {}^\alpha\alpha^{(Id)} \text{ and } u(j) = j \text{ for all } j \neq i\} = c_i L$. From $p(Id, r) \leq a$, (iii) readily follows.
- (iv) follows from (iii).
- (v) We first, like we did in the finite case, extend p to a BA automorphism p' , say, of $Rl_L A(\alpha)$. This is possible by (i). Then we define $f(x) = p'(x \cap L) \cup (x \cap -L)$. The rest is completely analogous to the proof of fact 4.2.

■

Theorem 4.5. $\mathfrak{B}(\alpha)$ is an elementary subalgebra of $\mathfrak{A}(\alpha)$.

Proof. This can be proved using the Tarski-Vaught test. □

Now we show that $B(\alpha)$ is not a neat reduct. We proceed basically as we did in the finite case, the trick being “a(n) (infinite) cardinality twist” that first order logic cannot witness.

Theorem 4.6. Let $\mathfrak{Rl}_L \mathfrak{B}(\alpha) = \{x \in B(\alpha) : x \leq L\}$. Then $|\mathfrak{Rl}_L \mathfrak{B}(\alpha)| \leq \alpha$.

Proof. First of all note that by 6.1 (i) we have $L \in B$. Now let

$$X = \cup \{P(u) : u \in S_\alpha - \{Id\}\} \cup P_\omega(Id),$$

$$Y = \{c_i a : a \in A\} \cup \{d_{ij} : i, j \in \alpha\}, \text{ and}$$

$$D(\alpha) = Sg^{BLA(\alpha)}(X \cup Y).$$

Then $|Y| = \alpha$. This can be seen by noting that for every $x \in Y$, if x is non zero, and x is not a diagonal element, then by Lemma 1*(ii) there is an S with $S \subseteq {}^\alpha\alpha^{(Id)}$ such that $x = \cup \{1_u : u \in S\}$. Also $D(\alpha)$ is a subuniverse of $A(\alpha)$ because of the following: $D(\alpha)$ is closed under the boolean set operations, contains the diagonals, and for $a \in D(\alpha)$, and $i < \alpha$, we have $a \in A(\alpha)$ and so $c_i a \in D(\alpha)$, that is $D(\alpha)$ is closed under cylindrifications. Since $X \subseteq D(\alpha)$, X generates $B(\alpha)$ and $D(\alpha) \in CA_\alpha$, we have $B(\alpha) \subseteq D(\alpha)$. Let $rl_L(a) = a \cap L$. Then we have

$$Rl_L D(\alpha) = \{x \in D(\alpha) : x \leq L\} = Sg^{BLA(\alpha)} rl_L(X \cup Y).$$

But $|rl_L(X \cup Y)| = \alpha$, because $P_\omega(Id)$ is countable and $|Y| = \alpha$. Therefore $|rl_L(X \cup Y)| = \alpha$, and so $|Rl_L D(\alpha)| \leq \alpha$. By $Rl_L B(\alpha) \subseteq Rl_L D(\alpha)$ we are done. ■

Theorem 4.7. *Let $p[0, 1]$ be the transposition on α interchanging 0 and 1. Let $v = p[0, 1] \circ Id = \langle 1023 \dots \rangle$. Then $v \in S_\alpha$ and the following hold:*

- (i) $t^{B(\alpha)}(1_v) = 1_{Id}$
- (ii) For all $\beta > \alpha$, $CA_\beta \models_\alpha s(0, 1)c_\alpha x \leq t(c_\alpha x)$.
- (iii) For all $\beta > \alpha$ and $A \in CA_\beta$, $_\alpha s(0, 1) \in Ism(BlNr_\alpha A, BlNr_\alpha A)$.

Theorem 4.8. $\mathfrak{B}(\alpha) \notin \mathfrak{Nr}_\alpha CA_{\alpha+1}$.

Proof. If $B(\alpha)$ were a neat reduct then we would have $|Rl_L(B(\alpha))| = |R|$. But this is a contradiction. ■

$\mathfrak{A}(\alpha)$ constructed so far may or may not be a neat reduct. We now impose further restrictions on the C_r 's (or rather on the structure $\mathbf{U} = \langle U, C_r \rangle$), described in Lemma 1*, that *force* $A(\alpha)$ to be in $\cap_{k \in \omega} Nr_\alpha Ws_{\alpha+k}$. These restrictions are described in Lemma 1** below. Forming a “limit” out of the $Ws_{\alpha+k}$'s ($k \in \omega$), the *neat* $-\alpha$ reduct of which is $A(\alpha)$, we will show that $A(\alpha) \in Nr_\alpha Ws_{\alpha+\beta}$ for infinite β as well. This should be done, of course, in such a way that does not interfere with what we have already established, namely:

$B(\alpha)$ is *not* a neat reduct.

$A(\alpha)$ and $B(\alpha)$ are elementary equivalent.

To formulate Lemma 1** we fix some needed notation:

Notation . Let $1 \leq k < \omega$. Then $S(\alpha, \alpha + k)$ or simply

$$S(\alpha, k) = \{i \in {}^\alpha(\alpha + k) : \alpha + k - 1 \in Rgi \text{ and } \{m \in \alpha : |\{i(m) \neq m\}| < \omega\}.$$

That is $S(\alpha, k) = \{i \in {}^\alpha(\alpha + k)^{(Id)} : \alpha + k - 1 \in Rgi\}$.

$Cof^+ R$ denotes the set of all non empty finite or cofinite subsets of R . Let C_r be an α -ary relation symbol for all $r \in R$.

For any $X \subseteq R$, X finite we define the infinitary formulas:

$$\eta(X) = \vee \{C_r(x_0, \dots x_j \dots)_{j < \alpha} r \in X\}, \quad \text{and}$$

$$\eta(R - X) = \wedge \{\neg C_r(x_0, \dots x_j \dots)_{j < \alpha} : r \in X\}.$$

$\eta(X)$ and $\eta(R - X)$ are *restricted* formulas, in the sense of [HMT2] sec 4.3. Satisfiability for such formulas (by α -ary sequences) are defined the usual way. Below we shall have occasion to deal with (infinitary) formulas that are not restricted. These however can be obtained from restricted ones using the

(logical interpretation of the) CA operations, i.e quantification on finitely many variables and equality. Now we are ready to formulate the infinite version of Lemma 1.

Theorem 4.9. *There are a set W , an equivalence relation E with α -many blocks on W , $s \in {}^\alpha W$ with $s_i \in W_i$ the i th block, $i \in \alpha$, and α -ary relations C_r on W for all $r \in R$, such that conditions (i)-(v) below hold:*

- (i) (Blocks) $C_r(w_0, w_1 \cdots w_i \cdots)_{i < \alpha}$ implies $D_E(w_0, w_1 \cdots w_i \cdots)_{i < \alpha}$, for all $r \in R$ and $w_i \in W$. Here $D_E(w_0, w_1 \cdots w_i \cdots)_{i < \alpha}$ means that for any $i < j < \alpha$, w_i, w_j are not E -equivalent i.e. they are in distinct blocks.
- (ii) (Symmetry) For all $f \in {}^\alpha W^{(s)}$ for all $r \in R$, for all permutations $\pi \in {}^\alpha \alpha^{(Id)}$, if $f \in C_r$ then $f \circ \pi \in C_r$.
- (iii) (Bigness) For all $r \in R$, for all $i < \alpha$ and $v \in {}^\alpha W^{(s)}$ such that $D[v_i]_{i < \alpha}$ there exists $x \in W$ such that $v[i|x] \in C_r$.
- (iv) (Saturation) For all $1 \leq k < \omega$, for all $v \in {}^{\alpha+k-1}W^{(s)}$ one to one, for all $x \in W$, for any function $g : S(\alpha, k) \rightarrow Cof^+R$ for which $\{i \in S(\alpha, k) : |\{g(i) \neq R\}| < \omega\}$, there is a $v_{\alpha+k-1} \in W \setminus Rgv$ such that $xEv_{\alpha+k-1}$, i.e. $v_{\alpha+k-1}$ is in the same block as x , and

$$\bigwedge \{D(v_{i_j})_{j < \alpha} \implies \eta(g(i))[\langle v_{i_j} \rangle] : i \in S(\alpha, k)\}.$$

- (v) (Disjointness) The C_r 's are pairwise disjoint.

Proof. Let $k \geq 1$. For brevity set

$$Y(\alpha, R, k) = \{g \in {}^{S(\alpha, k)}Cof^+R : \{i \in S(\alpha, k) : |\{g(i) \neq R\}| < \omega\}.$$

Let

$$Q = \cup \{I^{(\alpha+k-1)}|R| \times Y(\alpha, R, k) : 1 \leq k < \omega\},$$

then $|Q| = |R| = \kappa$, say. Here $I^{(\alpha+k-1)}|R|$ stands for the set of all one to one functions from $\alpha + k - 1$ into R . Let ρ be an enumeration of Q such that: for all $l < \kappa$, for all $q \in Q$, there exists j with $l < j < \kappa$ such that $\rho(j) = q$. Fix a well ordering \prec of R . let $l < \kappa$ and suppose that for all $\mu < l$ we have already defined the element x_μ , and the α -ary relation $C_r^\mu \subseteq {}^\alpha W_\mu^{(s)}$, where $W_\mu = \{x_k : k < \mu\}$. Assume that $\rho(l) = \langle \langle \beta_j \rangle_{j < \alpha+k-1}, f \rangle$. Then $\beta_j \in |R|$ for all $j < \alpha + k - 1$, and $f \in Y(\alpha, R, k)$. Let x_l be an element not in $\cup \{W_\mu : \mu < l\}$. If there exists $\mu < \alpha + k - 1$ such that $l \leq \beta_\mu$, then for all $r \in R$ we define $C_r^l = \cup \{C_r^\mu : \mu < l\}$. Else, $l > \beta_\mu$ for all $\mu < \alpha + k - 1$. Let $x_\mu = x_{\beta_\mu}$ for $\mu < \alpha + k - 1$ and let $x_{\alpha+k-1} = x_l$. For all $r \in R$ and $i \in S(\alpha, k)$ we let

$$\langle u_{i_j} \rangle_{j < \alpha} \in X_r^l \text{ iff } r \text{ is the } \prec - \text{ least element of } f(i)$$

and

$$C_r^l = \cup \{C_r^i : i < l\} \cup \{\langle t_{\pi(0)}, t_{\pi(1)}, \dots \rangle : t \in X_r^l \text{ and } \pi \in S_\alpha\}.$$

Finally set

$$W = \cup \{W_l : l < \mu\} \quad \text{and} \quad C_r = \cup \{C_r^l : l < \mu\}.$$

It is not hard to check just like we did in the finite dimensional case, that $\mathbf{W} = \langle W, C_r \rangle_{r \in R}$ is as desired. \blacksquare

Let $\mathbf{W} = \langle W, C_r \rangle$ and $s \in {}^\alpha W$ be as specified above. Then W is a disjoint union of α copies. We let W_i denote the i th copy. We shall construct our algebras having unit ${}^\alpha W^{(s)}$. Recall that S_α stands for the set of permutations in ${}^\alpha \alpha^{(Id)}$. Let $u \in S_\alpha$ and $r \in R$. Then we set

$$p(u, r) = C_r \cap W_{u_0} \times W_{u_1} \times \dots \times W_{u_i} \times \dots \cap {}^\alpha W^{(s)}.$$

$$P(u) = \{p(u, r) : r \in R\}.$$

Let N be a fixed countable subset of R . Let

$$P_\omega(u) = \{p(u, r) : r \in R\}.$$

With a slight abuse of notation we let $A(\alpha)$ and $B(\alpha)$ be the (new) algebras constructed out of the (new) structure $\mathbf{W} = \langle W, C_r \rangle_{r \in R}$, described in the previous lemma. That is $A(\alpha) \in Ws_\alpha$ and $B(\alpha) \subseteq A(\alpha)$ are defined as follows:

$$A(\alpha) = Sg^C(\cup \{P(u) : u \in S_\alpha\});$$

and

$$B(\alpha) = Sg^C((\cup \{P(u) : u \in S_\alpha \setminus \{Id\}\}) \cup P_\omega(Id)).$$

Here C is the full weak set algebra with greatest element ${}^\alpha W^{(s)}$, with s as specified in Lemma 1**. Recall that $s(i) \in W_i$, the i th copy. It is easy to check that $A(\alpha)$ and $B(\alpha)$, so defined, are (still) elementary equivalent (copy the proof of facts 6.1-2) and $B(\alpha)$ is (still) not a neat reduct (copy the proof of facts 6.3-6.5). Like in the finite case, we will use the new condition of “saturation” or “elimination of quantifiers” expressed in Lemma 1** (iv) to show that $A(\alpha) \in \cap_{k \in \omega} Nr_\alpha Ws_{\alpha+k}$. To “lift” $A(\alpha)$ to arbitrary finite extra dimensions, we shall further need:

Some more Definitions.

Let $0 \leq k < \omega$. Then we let $W_k = W = \cup_{i < \alpha} W_i$. Fix $w \in W$. Let $s_k = s \cup \{\langle i, w \rangle : \alpha \leq i < (\alpha + k)\}$. Then $s_k \in {}^{\alpha+k} W$. (When $k = 0$, then s_k is just s). Let V_k be the weak space ${}^{\alpha+k} W^{(s_k)}$ and let $C(\alpha + k)$ be the full $Ws_{\alpha+k}$ with unit V_k .

Let $u \in {}^{\alpha+k}\alpha^{(Id)}$. Then

$$F_k(u) = \{C_r(x_{i_0}, \dots x_{i_j} \dots)_{j < \alpha}, \neg C_r(x_{i_0}, \dots x_{i_j} \dots)_{j < \alpha} : r \in R, i \in {}^\alpha\alpha + k^{(Id)} \\ \text{and } \langle u_{i_0}, \dots u_{i_j} \dots \rangle_{j < \alpha} \text{ is one to one } \} \cup Eq(u).$$

Recall that $Eq(u)$ stands for $\{x_i = x_j : x_i \neq x_j : (i, j) \in \ker(u)\}$. Like the finite case we assume that $\{T, F\} \in F_k(u)$. We put

$$F_k(u)^* = \{\wedge J : J \subseteq_\omega F_k(u)\} \text{ and}$$

$$F_k(u)^{**} = \{\vee J : J \subseteq_\omega F_k(u)^*\}.$$

Let $u \in {}^{\alpha+k}\alpha^{(Id)}$ and $\phi \in F_k(u)^{**}$.

Then we let $E(u, \phi)$ denote the following $(\alpha + k)$ -ary relation on V_k

$$\{s \in V_k : s_j \in W_{u_j} \text{ for all } j < \alpha + k \text{ and } \mathbf{W} \models \phi[\langle s_j \rangle_{j < \alpha + k}]\}.$$

As noted before, satisfiability is defined the usual way. In particular, when $k = 0$, $u \in S_\alpha$ and $r \in R$, then $E(u, C_r(x_0, x_1, \dots))$ is the same as $p(u, r)$.

$ind(\phi)$ denotes the set $\{i \in \alpha + k : x_i \in var(\phi)\}$.

We let $G(k) = \{c_{(\Delta)}E(u, \phi), -c_{(\Delta)}E(u, \phi) : \Delta \subseteq_\omega \alpha + k, u \in {}^{\alpha+k}\alpha^{(Id)}, \phi \in F_k(u)^{**}, \Delta \cap ind(\phi) = \emptyset\} \cup \{d_{ij}^{V_k}, -d_{ij}^{V_k} : i, j \in \alpha + k\}$.

Here $c_{(\Delta)}$, the unary operation referred to as generalized cylindrification in [6]. That is $c_{\emptyset}x = x$ and if $\Delta = \{k_0, \dots k_{m-1}\}$ is a non-empty finite subset of $\alpha + k$, then $c_{(\Delta)}x = c_{k_0} \dots c_{k_{m-1}}x$.

For $i, j \in \alpha + k$, $d_{ij}^{V_k}$, or simply d_{ij} is the diagonal element $\{s \in V_k : s_i = s_j\}$. Forming finite intersections of elements in $G(k)$ we let

$$G(k)^* = \{\cap J : J \subseteq_\omega G(k)\}$$

and forming finite unions of elements in $G(k)^*$, we finally let

$$G(k)^{**} = \{\cup J : J \subseteq_\omega G(k)^*\}.$$

We will show that $G(k)^{**}$ is a $Ws_{\alpha+k}$ (with unit V_k), that $G(0)^{**} = Nr_\alpha G(k)^{**}$ for all $k \in \omega$, and finally that $G(0)^{**} = A(\alpha)$. This will show that $A(\alpha) \in \cap_{k \in \omega} INr_\alpha Ws_{\alpha+k}$. To prove that $G(k)^{**}$ is a $Ws_{\alpha+k}$ we shall need:

Fact 6.6 .

- (i) Let $u \in {}^{\alpha+k}\alpha^{(Id)}$, $\Delta \subseteq_\omega \alpha + k$ and $\phi \in F_k(u)^{**}$. Then $c_{(\Delta)}E(u, \phi) \in G(k)^{**}$.
- (ii) Let $g \in G(k)^{**}$ and $i, j \in \alpha + k$ be distinct. Then $s_i^j(g) = c_j(g \cap d_{ij}) \in G(k)^{**}$.

- (iii) Let $H(k) = \{g \in G(k) : g = c_{(\Delta)}E(u, \phi) : \Delta \subseteq_{\omega} \alpha + k, u \in {}^{\alpha+k}\alpha^{(Id)} \text{ and } \phi \in F_k(u)^{**}\}$. Then $H(k)$ is closed under finite intersections.

Proof.

- (i) We first prove (i) when Δ consists of one element j , say. That is we will show that for u and ϕ as indicated above and $j < \alpha + k$ we have $c_jE(u, \phi) \in G(k)^{**}$. We can and will assume that $\phi \in F_k(u)^*$ since

$$c_jE(u, \phi_1 \vee \cdots \phi_n) = c_jE(u, \phi_1) \cup \cdots c_jE(u, \phi_n)$$

and $G(k)^{**}$ is closed under finite unions. Then $\phi = \wedge J$, where $J \subseteq_{\omega} F_k(u)$. Next, we break up ϕ into two parts just like we did in the finite dimensional case; that is let $J^+ = \{\psi \in J : x_j \in \text{var}(\psi)\}$ and $J^- = \{\psi \in J : x_j \notin \text{var}(\psi)\}$. For brevity set $\phi^+ = \wedge J^+$ and $\phi^- = \wedge J^-$, where by the empty conjunction we understand the formula T . Now $\phi = \phi^+ \wedge \phi^-$, $\phi^- \in F_k(u)^*$ and x_j occurs only in ϕ^+ .

Notation . For $i, l \in \alpha + k$ and $\phi \in F_k(u)^{**}$ $\phi[i|l]$ stands for the formula obtained from ϕ by replacing all (free and bound) occurrences of x_i in ϕ by x_l .

Case (a): $x_j = x_m \notin J^+$ and $x_j = x_m \notin J^+$ for all $m \in \alpha + k - \{j\}$.

Then either $\mathbf{W} \models \phi^+ \equiv F$, in which case $\mathbf{W} \models \exists x_j \phi^+ \equiv F$, and so $c_jE(u, \phi) = c_jE(u, F) = 0 \in G(k)^{**}$. Else ϕ^+ is satisfiable, in which case, by applying the saturation condition described in Lemma 1**, we get

$$\mathbf{W} \models (\exists x_{\alpha+k-1} \phi^+[j|\alpha+k-1]) \equiv T,$$

which is the same as $\mathbf{W} \models (\exists x_j \phi^+) \equiv T$, and so

$$c_jE(u, \phi) = c_jE(u, (\exists x_j \phi^+) \wedge \phi^-) = c_jE(u, \phi^-).$$

The latter is in $G(k)^{**}$, since now x_j does not occur in ϕ^- , and $\phi^- \in F_k(u)^*$. Note that $\phi^- \in F_k(v)^*$ for any v that differs from u in at most the j -th place; and for any such v we have $c_jE(u, \phi) = c_jE(v, \phi^-)$.

Case (b): $x_j = x_m \in J^+$ or $x_m = x_j \in J^+$ for some $m \in \alpha + k$ and $m \neq j$.

Assume that $x_j = x_m \in J^+$. Then, by definition of $F_k(u)$, we have $u_j = u_m$. Also $\phi = (\wedge I) \wedge x_j = x_m$, for some $I \subseteq_{\omega} F_k(u)$. For brevity let $\psi = \wedge I$. Then of course $\psi \in F_k(u)^*$. Computing we get

$$c_jE(u, \phi) = c_jE(u, \psi \wedge x_j = x_m) = c_jE(u, \psi[j|m]).$$

The latter is in $G(k)^{**}$ since x_j does not occur in $\psi[j|m]$, and $\psi[j|m] \in F_k(u)^*$ because $\psi \in F_k(u)^*$ and $u_j = u_m$. In fact, it is not hard to see that $\psi[j|m] \in F_k(v)^*$, for any v that differs from u in at most the j th place. The case when Δ is an arbitrary finite subset of $\alpha + k$, follows by using the above reasoning eliminating the variables occuring in ϕ , with indices in Δ , one by one. Alternatively one can use a straightforward induction on $|\Delta|$. Note that we have actually proved the following infinite analogue of $(**)$ in fact 3.1 way above:

(*)** . Let $u \in {}^\alpha\alpha + k^{(Id)}$ and $\phi \in F_k(u)^{**}$. Let $\Delta = \{i_0, \dots, i_{n-1}\}$ be a finite subset of $\alpha + k$. Let $v \in {}^\alpha\alpha + k^{(Id)}$ be such that $v(j) = u(j)$ for all $j \notin \Delta$. Then there exists $\psi \in F_k(v)^{**}$ such that $\text{var}(\psi) \cap \{x_{i_0}, \dots, x_{i_{n-1}}\} = \emptyset$. i.e., $\text{ind}(\psi) \cap \Delta = \emptyset$, and $\mathbf{W} \models \psi \equiv \exists x_{i_0} \dots \exists x_{i_{n-1}} \phi$. In particular, $c_{(\Delta)}E(u, \phi) = c_{(\Delta)}E(v, \psi)$.

(ii) Let $g \in G(k)^{**}$ and $i, j \in \alpha + k$ be distinct.

Case (a): $g = c_{(\Delta)}E(u, \phi)$ with $u \in {}^{\alpha+k}\alpha^{(Id)}$, $\phi \in F_k(u)^{**}$.

Subcase (a) : $j \in \Delta$.

Then by [6] 1.5.8 (i) and 1.7.3 we have

$$s_i^j(g) = s_i^j c_j(c_{(\Delta - \{j\})})E(u, \phi) = c_j(c_{(\Delta - \{j\})})E(u, \phi) = g$$

Subcase (b) : $j \notin \Delta$ and $i \notin \Delta$.

Then by [6], 1.5.8 (ii) we have

$$s_i^j g = s_i^j c_{(\Delta)}E(u, \phi) = c_{(\Delta)}s_i^j E(u, \phi).$$

Let $g^* = E(u, \phi) \cap d_{ij}$. Then g^* and hence $s_j^i g = c_{(\Delta)}c_j g^*$, is equal to zero if $u_i \neq u_j$. Else, as easily checked, $g^* = E(u, \phi \wedge x_i = x_j)$. By $u_i = u_j$ and $\phi \in F_k(u)^{**}$, we have $\phi \wedge x_i = x_j \in F_k(u)^{**}$. By fact 6.6 (i) we get that

$$s_j^i g = c_{(\Delta)}c_j g^* = c_{(\Delta)}c_j E(u, \phi \wedge x_i = x_j) \in G(k)^{**}.$$

Subcase (c) : $j \notin \Delta$ and $i \in \Delta$.

By [6] 1.7.3 and 1.5.8 (ii) we have

$$s_i^j c_{(\Delta)}E(u, \phi) = s_i^j c_{(\Delta - \{i\})}c_i E(u, \phi) = c_{(\Delta - \{i\})}s_i^j c_i E(u, \phi).$$

Let $g^* = s_i^j c_i E(u, \phi)$. Then by [6] 1.5.1 we have $g^* = c_j(c_i(E(u, \phi) \cap d_{ij}))$. Let $v = u_{u(j)}^i = u \circ [i|j]$. Here and elsewhere $[i|j]$ denotes the replacement

that maps i to j and otherwise coincides with the identity. Then $v(j) = u(j) = v(i)$. Also, it is easy to see that

$$c_i(E(u, \phi) \cap d_{ij}) = E(v, \phi \wedge x_i = x_j).$$

Since $i \in \Delta$, we can assume by fact 6.6 (i) that x_i does not occur in ϕ . Since v differs from u in at most the i th place, we have $\phi \in F_k(v)^{**}$. But $v(i) = v(j)$, and so $x_i = x_j \in F_k(v)^{**}$, thus $\phi \wedge x_i = x_j \in F_k(v)^{**}$. By fact 6.6 (i) we get that g^* hence g is in $G(k)^{**}$.

Case (b) : $g = d_{kl}$, where $k, l \in \alpha + k$.

By [6], 1.5.4 we have $s_i^k(d_{kl}) = d_{il}$ if $k \neq l$ and $s_i^j d_{kl} = d_{kl}$ if $j \notin \{k, l\}$. In either case we have $s_i^j g \in G(k)^{**}$.

Case (c) : g is an element in $G(k)^{**}$.

Follows from the two previous cases, since the s_i^j 's ($i, j \in \alpha + k$) are BA endomorphisms of the boolean algebra $G(k)^{**}$ (cf. [6] 1.5.3 (i)).

- (iii) Let $g = c_{(\Delta_1)}E(u_1, \phi_1) \cap c_{(\Delta_2)}E(u_2, \phi_2)$ where $\Delta_i \cap \text{ind}(\phi_i) = 0$ for $i \in \{1, 2\}$. Then $g = 0$ if there exists $j \in (\alpha + k) - (\Delta_1 \cup \Delta_2)$ such that $u_1(j) \neq u_2(j)$. Else, it is not difficult to check that $g = c_{(\Delta_1 \cap \Delta_2)}E(w, \phi_1 \wedge \phi_2)$, where w is defined as follows:

$$w(j) = u_1(j) = u_2(j) \text{ for } j \in \alpha + k - (\Delta_1 \cup \Delta_2)$$

$$w(j) = u_1(j) \text{ if } j \in \Delta_2 - \Delta_1$$

$$w(j) = u_2(j) \text{ if } j \in \Delta_1 - \Delta_2, \text{ and } w \text{ is defined arbitrarily on } \Delta_1 \cap \Delta_2.$$

Also by $\text{ind}(\phi_i) \cap \Delta_i = 0$, for $i \in \{1, 2\}$ and $w(j) = u_1(j) = u_2(j)$ for all $j \in \alpha + k - (\Delta_1 \cup \Delta_2)$ we can assume by 6.6 (i) that $\phi_i \in F_k(w)^{**}$, hence $\phi_1 \wedge \phi_2 \in F_k(w)^{**}$.

■

To economise on notation we recall a few notions from [6]. A *generalized diagonal element* is a finite intersection of diagonal elements. A *co-diagonal element* is the complement of a diagonal element. A *generalized co-diagonal element* is a finite intersection of co-diagonal elements. In particular the unit of an algebra can be viewed as both a generalized diagonal and a generalized co-diagonal.

Fact 6.7 . For all $0 \leq k < \omega$, $G(k)^{**} \in Ws_{\alpha+k}$.

Proof. Clearly $G(k)^{**}$ is a boolean field of sets with greatest element V_k . Also $G(k)^{**}$, by definition, contains all diagonal elements. We are thus left to check cylindrifications, by the additivity of which, it suffices to show that for every

$g \in G(k)^*$ and $j < \alpha + k$ we have $c_j g \in G(k)^{**}$. By fact 6.6 (iii) a typical element g of $G(k)^*$ is of the form

$a_0 \cap b \cap c \cap -a_1 \cap -a_2 \cdots \cap -a_n$, where $n \in \omega$, $a_0, a_1, \dots, a_n \in H(k)$, b is a generalized diagonal element, and c is a generalized co-diagonal element. Now fix $j < \alpha + k$ and let g be as indicated above.

Case (a) : $b = d_{jl} \cap b'$ where b' is a generalized diagonal element and $j \neq l$. Let $g^* = a_0 \cap b' \cap c \cap -a_1 \cdots \cap -a_n$. Then, of course $g^* \in G(k)^*$ and $c_j g = s_l^j g^*$. By fact 6.6 (i) the latter is in $G(k)^{**}$.

Case (b) : $a_0 = b = V_k$.

In this case we have $g = -c_{(\Delta_1)} E(u_1, \phi_1) \cdots \cap -c_{(\Delta_n)} E(u_n, \phi_n) \cap -d_{jl_1} \cdots \cap -d_{jl_m}$, where $n, m \in \omega$, $\Delta_k \subseteq_\omega \alpha + k$ for $1 \leq k \leq n$ and since $c_j(x \cap c_j y) = c_j x \cap c_j y$, we can assume that $j \notin \Delta_1 \cup \cdots \cup \Delta_n$. Assume that $g \neq 0$, for else there is nothing more to prove. In particular, j, l_1, \dots, l_m are pairwise distinct. We claim that in this case we have $c_j g = V_k$.

Indeed, let $s = \langle s_l \rangle_{l \in \alpha + k}$ be an arbitrary element in V_k . We will show that $s \in c_j g$, by which we will be done. Choose $k_j \notin \{u_1(j), \dots, u_n(j)\} \cup \{i_{l_1}, \dots, i_{l_m}\}$, and let u be an arbitrary element in W_{k_j} . Let $z = s(j|u)$. Assume that $1 \leq i \leq n$. Then $z \notin c_{(\Delta_i)} E(u_i, \phi_i)$ because $z(j) = s(j|u)(j) = u \notin \cup_{i < n} W_{u_i(j)}$. Assume now that $1 \leq i \leq m$. Then $z(j) \neq z(l_i)$, since they are in distinct copies by the choice of k_j , i.e. $z \notin d_{jl_i}$. We have shown that $z \in g$. Since s differs from z in at most the j th place we get that $s \in c_j g$. We have proved that $c_j g = V_k$.

Case (c) :

$$g = c_{(\Delta_0)} E(u_0, \phi_0) \cap -c_{(\Delta_1)} E(u_1, \phi) \cap \cdots \cap c_{(\Delta_n)} E(u_n, \phi_n) \cap -d_{jl_1} \cdots \cap -d_{jl_m},$$

where $n, m \in \omega$, $\Delta_k \subseteq_\omega \alpha + k$ for $0 \leq k \leq n$ and as in the previous case, we can assume that $j \notin \Delta_0 \cup \cdots \cup \Delta_n$ and that j, l_1, \dots, l_m are pairwise distinct.

We can further assume that $\{l_1, \dots, l_m\} \subseteq \Delta_0$, for if $l_k \notin \Delta_0$ for some $1 \leq k \leq n$, then we can simply “ignore” the co-diagonal element with indices j and l_k , i.e. the element $-d_{jl_k}$, because of the following:

$$c_{(\Delta_0)} E(u_0, \phi_0) \cap -d_{jl_k} = c_{(\Delta_0)} E(u_0, \phi_0 \wedge x_j \neq x_{l_k})$$

if $u_0(j) = u_0(l_k)$, and is equal to $c_{(\Delta_0)} E(u_0, \phi_0)$ otherwise. Note too, that $\phi_0 \wedge x_j \neq x_{l_k} \in F_k(u_0)^{**}$, whenever $\phi_0 \in F_k(u_0)^{**}$ and $u_0(j) = u_0(l_k)$. In either case, g can be written in the form:

$$c_{(\Delta_0)} E(u_0, \psi) \cap \cdots \cap c_{(\Delta_n)} E(u_n, \phi_n) \cap \cap \{-d_{jl} : l \in \{l_1, \dots, l_m\} - l_k\},$$

where $c_{(\Delta_0)} E(u_0, \psi) \in H(k)$.

Now let

$$\Gamma = \Delta_0 - \{l_1, \dots, l_m\} \quad \text{and} \quad \Gamma_1 = \{l_1, \dots, l_m\}.$$

Then $\Delta_0 = \Gamma \cup \Gamma_1$. Let $S = \{j\} \times \Gamma_1 = \{(j, l) : l \in \Gamma_1\}$ and let d_S be the co-diagonal element: $\cap\{-d_{jl} : (j, l) \in S\}$.

Let $g^* = -c_{(\Delta_1)}E(u_i, \phi_1) \cap \dots - c_{(\Delta_n)}E(u_n, \phi_n)$, $g_1 = c_{(\Gamma)}E(u_0, \phi_0) \cap g^*$ and $g_2 = c_{(\Gamma_1)}E(u_0, \phi_0) \cap d_S \cap g^*$. Then by [6] 1.7.3, and the additivity of c_j we have $c_j g = c_j g_1 \cup c_j g_2$.

We shall show that $c_j g_1 \in G(k)^{**}$ and $c_j g_2 \in G(k)^{**}$, by which we will be done.

*Proof of $c_j g_1 \in G(k)^{**}$.*

Write $g_1 = c_{(\Gamma)}E(u_0, \phi_0) \cap \dots - c_{(\Delta_n)}E(u_n, \phi_n)$.

We can assume that for $1 \leq k \leq n$ and $i \notin \Gamma \cup \Delta_k$, we have $u_0(i) = u_k(i)$, for if $u_0(i) \neq u_k(i)$ for some $1 \leq k \leq n$, and some $i \notin \Gamma \cup \Delta_k$, then $c_{(\Gamma)}E(u_0, \phi_0) \cap -c_{(\Delta_k)}E(u_k, \phi_k) = c_{(\Gamma)}E(u_0, \phi_0)$.

For the time being assume that $\phi_1 = \phi_2 = \dots \phi_n = T$.

Claim

$$c_j g_1 = c_j c_{(\Gamma)}E(u_0, \phi_0) \cap -c_j c_{(\Delta_1)}E(u_1, T) \cap \dots - c_j c_{(\Delta_n)}E(u, T).$$

Proof of claim

It is straightforward to show that r.h.s is contained in $c_j g$. Now for the opposite inclusion:

Let $s \in c_j g_1$. Then there exists $b \in W$ such that $s(j|b) \in c_{(\Gamma)}E(u_0, \phi_0)$ and $s(j|b) \notin c_{(\Delta_k)}E(u_k, T)$ for all $1 \leq k \leq n$. Thus $s \in c_j c_{(\Gamma)}E(u_0, \phi_0)$. We will show that $s \notin c_j c_{(\Delta_k)}E(u_k, T)$ for all $1 \leq k \leq n$, by which case we will be done. Towards this end, fix $1 \leq k \leq n$ and assume to the contrary that $s \in c_j c_{(\Delta_k)}E(u_k, T)$. Then there exists $c \in W$ such that $s(j|c) \in c_{(\Delta_k)}E(u_k, T)$. By $s_c^j \in c_{(\Delta_k)}E(u_k, T)$ and $j \notin \Delta_k$ we get $c = u_k(j)$. Similarly, by $s_b^j \in c_{(\Gamma)}E(u_0, \phi_0)$ and $j \notin \Gamma$ we get $b = u_0(j)$. Recall that we assumed that $u_0(j) = u_k(j)$ since $j \notin \Gamma \cup \Delta_k$, thus $b = c$. Since $s(j|b)$ and $s(j|c)$ differ in at most the j th place, $b = c$ and $s(j|c) \in c_{(\Delta_k)}E(u_k, T)$ we get $s(j|b) \in c_{(\Delta_k)}E(u_k, T)$. Contradiction proving that s is as required.

Now the general case follows from this by noting that

$$-c_{(\Delta)}E(u, \phi) = c_{(\Delta)}E(u, \neg\phi) \cup -c_{(\Delta)}E(u, T),$$

and that $H(k)$ is closed under finite intersections.

We now turn to showing that $c_j g_2 \in G(k)^{**}$. This will be done by showing that $a = c_{(\Gamma_1)}E(u_0, \phi_0) \cap d_S$, hence $g_2 = a \cap g^*$, can be written as a finite union of elements of the form $a_0 \cap -a_1 \cap \dots \cap -a_n$ where $a_i \in H(k)$. Then by the additivity of c_j , and the previous case we get the desired. We start off when Γ_1 consists of a single element m say. We compute $a = c_m E(u_0, \phi_0) \cap -d_{jm}$. For brevity set $u = u_0$. Let $v = u(m|u(j)) = u \circ [m|j]$. Then $v(m) = v(j) = u(j)$. Moreover it is not hard to check that

$$a = (c_m E(u, \phi_0) \cap -E(v, T)) \cup E(v, \phi_0 \wedge x_j \neq x_m).$$

Since x_m does not occur in ϕ_0 , and v differs from u in at most the m -th place, we infer from fact 6.6 (i) that $\phi_0 \in F_k(v)^{**}$. By $v(m) = v(j)$ we get that $x_j \neq x_m$

is also in $F_k(v)$ hence in $F_k(v)^{**}$. It follows that $\phi_0 \wedge x_j \neq x_m \in F_k(v)^{**}$. We have shown that $E(v, \phi_0 \wedge x_j \neq x_m) \in H(k)$. Thus a is as desired, since $c_m E(u, \phi_0)$ and $E(v, T)$ are in $H(k)$, too. Now for the general case. Assume that $\Gamma_1 = \{l_1, \dots, l_m\}$. Having treated the case when $m = 1$ we now assume that $m > 1$. Then

$$a = c_{(\Gamma_1)} E(u, \phi) \cap -d_S = \cap \{a_k : 1 \leq k \leq m\}$$

where $a_k = c_{(\Gamma_1 - \{l_k\})} c_{l_k} E(u, \phi) \cap -d_{j_{l_k}}$. Therefore a , hence $g_2 = a \cap g^*$, is indeed the finite union of elements of the $a_0 \cap -a_1 \cdots \cap -a_n$ where $a_i \in H(k)$. Thus $c_j g_2$ is in $G(k)^{**}$. By this the proof of fact 6.7 is complete. ■

Fact 6.8 . $G(0)^{**} \cong Nr_\alpha G(k)^{**}$, for all $0 \leq k < \omega$.²

Proof. Let $k \geq 0$. Define $i(k)$ like in fact 3.3. That is for $a \in G(0)^{**}$, let $i(k)(a) = \{t \in V_k : t \upharpoonright \alpha \in a\}$. Then $i(k) \in Ism(A(\alpha), Nr_\alpha C(\alpha + k))$, where $C(\alpha + k)$ is the full $Ws_{\alpha+k}$ with unit V_k . We will show that $i(k)G(0)^{**} \subseteq G(k)^{**}$. For that it clearly suffices to show that for all $u \in {}^\alpha \alpha^{(Id)}$, and $\phi \in F_0(u)$, we have $i(k)E(u, \phi) \in G(k)^{**}$. Let u and ϕ be as indicated: Then $i(k)E(u, \phi) = \cup \{E(v, \phi) : v \in {}^{\alpha+k} \alpha^{(Id)} : v \upharpoonright \alpha = u\} = c_\alpha \cdots c_{\alpha+k-1} E(u^*, \phi)$ where $u^* = u \cup \{ \langle i, 0 \rangle : \alpha \leq i < \alpha + k \}$. Since $F_0(u) \subseteq F_k(v)$ whenever $v \in {}^{\alpha+k} \alpha^{(Id)}$ is such that $v \upharpoonright \alpha = u$, we get $\phi \in F_k(u^*)^{**}$, hence $i(k)E(u, \phi) \in G(k)^{**}$. We have shown that $i(k) \in Ism(G(0)^{**}, Nr_\alpha G(k)^{**})$. We will now show that $i(k)$ is actually onto $Nr_\alpha G(k)^{**}$. Since the c_i 's are additive, it suffices to show that for all $g \in G(k)^*$, there exists $a \in G(0)^{**}$, such that $i(k)a = c_\alpha \cdots c_{\alpha+k-1}g$.

Case (a): $g = E(v, \phi)$, $v \in {}^{\alpha+k} \alpha^{(Id)}$, and $\phi \in F_k(v)^{**}$.

Then by fact 6.6 (i) there exists $\psi \in F_k(v)^*$ such that $var(\psi) \cap \{x_\alpha, \dots, x_{\alpha+k-1}\} = \emptyset$ and $\mathbf{W} \models \exists x_\alpha \cdots \exists x_{\alpha+k-1} \phi \equiv \psi$. Let $u = v \upharpoonright \alpha \in {}^\alpha \alpha^{(Id)}$. By noting that $F_0(u)^* = \{\phi \in F_k(v)^* : var(\phi) \subseteq \alpha\}$, we get that $\psi \in F_0(u)^*$. Moreover we have $i(k)E(u, \psi) = c_\alpha \cdots c_{\alpha+k-1} E(u, \phi)$.

Case (b): $g = a \cap c$, where $a = -a_0 \cap \cdots \cap -a_n$, $a_i \in H(k)$ and c is a generalized codiagonal.

If $c_\alpha \cdots c_{\alpha+k-1}g = g$, then $g \in G(0)^{**}$. Else $c_\alpha \cdots c_{\alpha+k-1}g$ is either 0 or V_k , pending on whether $g = 0$ or not. The choice of $a \in G(0)^{**}$ in this case is also obvious.

Case (c): $g = d_0 \cap -d_1 \cap \cdots \cap -d_n \cap b \cap c$, where $d_i \in H(k)$, b is a generalized diagonal, and c a generalized codiagonal.

Assume that $1 \leq k < \omega$; else there is nothing to prove.

Let $\Gamma = \{\alpha, \dots, \alpha + k - 1\}$. By the proof of fact 6.7, we have $c_{(\Gamma)}g$ is the finite union of elements of the form $a_0 \cap -a_1 \cdots \cap -a_n \cap b \cap c$ where $a_0, \dots, a_n \in H(k)$,

²We should point out that the complete analogue of fact 3.3 is true: That is for $0 \leq l < k < \omega$ we have $G(l)^{**} \cong Nr_{\alpha+l} G(k)^{**}$. Fact 6.8 is the special case when $l = 0$, which is all what we need to show that $\mathbf{A}(\alpha) \in Nr_\alpha Ws_{\alpha+k}$, for all $k < \omega$.

b is a generalized diagonal and c a generalized codiagonal. Further we can assume that if $b = \cap\{d_{ij}^{V_k} : i, j \in \Delta_1\}$ and $c = \cap\{-d_{ij}^{V_k} : i, j \in \Delta_2, i \neq j\}$ then $\Delta_i \subseteq_\omega \alpha$, for $i \in \{1, 2\}$. By the additivity of generalized cylindrifications (namely of c_Γ), we can and will assume that

$$c_\Gamma g = c_{(\Gamma_0)} E(v_0, \phi_0) \cap c_{(\Gamma_1)} E(v_1, \phi_1) \cdots \cap c_{(\Gamma_n)} E(v_n, \phi_n) \cap b \cap c,$$

where for each $i \leq n$ we have $\Gamma \subseteq \Gamma_i \subseteq_\omega \alpha + k$, $\phi_i \in F_k(v_i)^{**}$, b is a generalized diagonal and c is a generalized codiagonal as indicated above. By fact 6.6 (i) we can assume that $\text{var}(\phi_i) \cap \{x_\alpha, \dots, x_{\alpha+k-1}\} = \emptyset$. For $i \leq n$ let $u_i = v_i \upharpoonright \alpha$. Then $u_i \in {}^\alpha\alpha^{(Id)}$, and by fact 6.6 (i) $\phi_i \in F_0(u_i)^{**}$. Let $b' = \cap\{d_{ij}^{V_0} : i, j \in \Delta_1\}$ and $c' = \cap\{d_{ij}^{V_0} : i, j \in \Delta_2, i \neq j\}$.

Let $a = c_{(\Gamma_0-\Gamma)} E(u_0, \phi_0) \cap \cdots \cap c_{(\Gamma_n-\Gamma)} E(u_n, \phi_n) \cap b' \cap c'$. Then $a \in G(0)^{**}$ and an easy checked $i(k)a = c_\Gamma g$. \blacksquare

Fact 6.9 . $G(0)^* = A(\alpha)$.

Proof. Let $u \in {}^\alpha\alpha^{(Id)}$ be one to one and $r \in R$. Then by definition we have

$$p(u, r) = E(u, C_r(x_0, x_1, \dots)).$$

Hence $p(u, r) \in F_0(u)$. Since $G(0)^{**}$ is a Ws_α containing the generators of $A(\alpha)$ it follows that $A(\alpha) \subseteq G(0)^{**}$. For the other inclusion it is enough to show that for $u \in {}^\alpha\alpha^{(Id)}$ and $\phi \in F_0(u)$, we have $E(u, \phi) \in A(\alpha)$. We start by showing that $E(u, T) \in A(\alpha)$ for all such u .

Case (a): u is one to one.

Let $r \in R$. Then $c_0(p(u, r)) \cap c_1(p(u, r)) = E(u, T) \in A(\alpha)$.

Case (b): u is not one to one.

Let J be a finite subset of α , such that $u(J) \subseteq J$ and $u(i) = i$ whenever $i \notin J$. Such a J exists, but of course is not unique. Fix one such J . Let $u' = u \upharpoonright J$. Then $u' : J \rightarrow J$. Let $n = |Rgu'|$. Then $n < |J|$, because u hence u' is not one to one. We next proceed as in the finite dimensional case; correlating a permutation v to u' as follows: Let $y \in {}^n J$ such that $y(0) < \cdots < y(n-1)$ and $\{u'_{y(0)}, \dots, u'_{y(n-1)}\} = Rgu'$.

For $i < l$ put $m_{y(i)} = \min(u'^{-1}(u'_{y(i)}))$. Choose $v \in S_J$ (a permutation on J) such that $v(m_{y(i)}) = u'_{y(i)}$. Such a v exists. Let $I = J - \{m_{y(0)}, \dots, m_{y(l-1)}\}$; suppose that $I = \{j_0, \dots, j_k\}$, $k \in \omega$ and $j_0 < j_1 < \cdots < j_k$. Let $t_i^l(x) = c_l x \cap d_{il}$. Let $v* = v \cup Id_{\alpha-J}$. Then $v*$ is a permutation on α , and like the finite case, we have

$$t_{j(k)}^{u_{j(k)}} \circ \cdots \circ t_{j(0)}^{u_{j(0)}} E(v*, T) = E(u, T).$$

From case (a) we get that $E(u, T) \in A(\alpha)$. Now let $u \in {}^\alpha\alpha^{(Id)}$ and $i, j \in \ker(u)$. Then $E(u, x_i = x_j) = E(u, T) \cap d_{ij} \in A(\alpha)$. Also for all $u, v \in S_\alpha$,

we have by Lemma 1**(i), $E(u, C_r(x_0, x_1, \dots)) = E(u, C_r(x_{v(0)}, \dots)) = p(u, r)$. Finally by noting that $E(u, \neg\phi) = E(u, T) - E(u, \phi)$ we get the desired. ■

Fact 6.10 . $A(\alpha) \in INr_\alpha Ws_\beta$, for all $\beta > \alpha$.

Proof. By fact 6.9, it suffices to show that $A(\alpha) \in Nr_\alpha Ws_{\alpha+\beta}$ for infinite β . So let $\beta \geq \omega$. Let $V_0 = {}^\alpha W^{(s)}$ be the unit of $A(\alpha)$.

Let $w \in W$. Let $s_\beta = s \cup \{(i, w) : \alpha \leq i < \beta\}$. Then $s_\beta \in {}^\beta W$. Let V_β be the weak space ${}^{\alpha+\beta} W^{(s_\beta)}$ and let $C(\alpha+\beta)$, or simply $C(\beta)$, be the full $Ws_{\alpha+\beta}$ with unit V_β . For $k < \omega$ let $i(k, \beta)$ be the function with domain $G(k)^{**}$, such that $i(k, \beta)a = \{s \in V_\beta : s \upharpoonright \alpha + k \in a\}$, $a \in G(k)^{**}$. Then $i(k, \beta) : G(k)^{**} \rightarrow C(\beta)$. Let $G(\beta) = Sg^{C(\beta)}\{i(k, \beta)G(k)^{**} : 0 \leq k < \omega\}$. Then $G(\beta) \in Ws_{\alpha+\beta}$. Moreover, it is easy to show that $G(\beta) = \cup\{i(k, \beta)G(k)^{**} : 0 \leq k < \omega\}$. From this together with facts 6.8 and 6.9, we get (using the same ideas of the proof of fact 3.4) that $A(\alpha) \cong Nr_\alpha G(\beta)$, thus $A(\alpha) \in INr_\alpha Ws_{\alpha+\beta}$. ■

Finally we show

Theorem 4.10. *The atom structure $\text{At}\mathfrak{B}$ is not weakly neat. That is for any atomic $\mathfrak{D} \in \mathbf{CA}_\alpha$ if $\text{At}\mathfrak{D} \cong \text{At}\mathfrak{B}$, then $\mathfrak{D} \notin \mathfrak{Nr}_\alpha \mathbf{CA}_{\alpha+1}$.*

Proof. Let \mathfrak{D} be an atomic algebra such that $\text{At}\mathfrak{D} \cong \text{At}\mathfrak{B}$. Then \mathfrak{D} will contain the term algebra which will be the disjoint copies indexed by ${}^\alpha \alpha^{(Id)}$ except that P_{Id} is countable. The same cardinality trick implemented above works. Had it been a neat reduct that ${}_\alpha s(0, 1)P_r = P_{Id}$ would be uncountable and this cannot happen. □

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